Nonforking in Short and Tame Abstract Elementary Classes

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Abstract

We develop a notion of nonforking for Abstract Elementary Classes. Under hypotheses of tameness, type shortness, few models, and extension, this nonforking is well behaved. We give an application to local character and the uniqueness of limit models.

Abstract Elementary Classes

$K = (K, \prec)$ is an Abstract Elementary Class (AEC) when K is a class of L structures and

1. $\prec$ is a partial order on K
2. if $\langle M_i \in K : i < \alpha \rangle$ is a increasing chain, then
   - $\cup_{i<\alpha} M_i \in K$ and, for all $i < \alpha$, we have $M_i \prec \cup_{i<\alpha} M_i$; and
   - if there is some $N \in K$ so that, for all $i < \alpha$, we have $M_i \prec N$, then we also have $\cup_{i<\alpha} M_i \prec N$.; and
3. $(K, \prec)$ respects L isomorphisms
4. for every $M, N \in K$, if $M \prec_K N$, then $M \subseteq_L N$
5. if $M_0, M_1, M_2 \in K$ with $M_0 \prec M_2$, $M_1 \prec M_2$, and $M_0 \subseteq M_1$, then $M_0 \prec M_1$;
6. $LS(K)$ is the first infinite cardinal $\lambda \geq |L(K)|$ such that for any $M \in K$ and $A \subset |M|$, there is some $N \prec M$ such that $A \subset |N|$ and $\|N\| \leq |A| + \lambda$.

Galois Types

Let A, B be sets and $M \in K$ a model. Then A and B have the same Galois type over M iff there is $f \in Aut_M \mathfrak{C}$ so $f(A) = B$. We denote this type $gtp(A/M)$.
Given $p = gtp(\langle a_i : i \in I \rangle A/M)$ and $I_0 \subset I$ and $N \prec M$, we define the restrictions

$$p^{I_0} \upharpoonright N = gtp(\langle a_i : i \in I_0 \rangle / N)$$

Clearly, if $p = q$, then every restriction is equal. We investigate AECs where the converse holds as well.
K is **short and tame** if there is a cardinal κ so that, for any $p, q \in S(M)$, we have

$$p = q \iff p^{I_0} \upharpoonright N = q^{I_0} \upharpoonright N \text{ for all } I_0 \subset I, N \prec M \text{ of size } < \kappa$$

Nonforking

Let A be a set and $M \prec N$ be models. Then $gtp(A/N)$ does fork over M, written $A \downarrow_M N$

Given any $a \in A$ and $N^- \prec N$ of size $< \kappa$, we have $gtp(a/N^-)$ is realized in M

This could also be called *coheir* or $<\kappa$ *satisfiability*.

This definition immediately satisfies **Invaraiance**, **Monotonicity**, **Transitivity**, and $< \kappa$ **Continuity**. We also want it to satisfy the following properties:

- **Existence**: Let A be a set and M_0 and N be models so that $M_0 \prec N$. Then there is some A' so that $gtp(A'/M_0) = gtp(A/M_0)$ and $A' \downarrow_{M_0} N$.
- **Symmetry**: Let A_1 be a set, M_0 be a model, and A_2 be a set so that there is a model M_2 with $M_0 \prec M_2$ and $A_2 \subset |M_2|$ so that $A_1 \downarrow_{M_0} M_2$. Then there is a model $M_1 \succ M_0$ that contains A_1 so that $A_2 \downarrow_{M_0} M_1$.
- **Uniqueness**: Let A and A' be sets and $M_0 \prec N$ be models. If $tp(A/M_0) = tp(A'/M_0)$ and $A \downarrow_{M_0} N$ and $A' \downarrow_{M_0} N$, then $gtp(A/N) = gtp(A'/N)$.

Main Theorem

Let K be an AEC with amalgamation, joint embedding, and no maximal models. If there is some $\kappa > LS(K)$ so that

1. K doesn't have the weak κ-order property
2. K is short and tame
3. $\downarrow$ satisfies existence

Then $\downarrow$ is an independence relation (see [1]).

Discussion of the Assumptions

A monster model

Amalgamation, joint embedding, and no maximal models are a common set of assumptions when working with AECs that appear often in the literature. They imply the existence of the monster model $\mathfrak{C}$.

Order Property

K has the *weak κ-order property* iff there are types $p \neq q \in S(M)$ such that, for all κ-like linear orders I, there is $\langle a_i b_i : i \in I \rangle$ so that, for all $i, j \in I$,

$$i \leq j \implies gtp(a_i b_j / M) = p$$
$$i > j \implies gtp(a_i b_j / M) = q$$

If κ is inaccessbile (or other methods), Shelah [3] has results that show this implies many models.

Short and Tame

These assumptions say that every Galois type is equivalent to its small approximations.
These are implied by the existence of an independence relation as we have.

Existence

This assumption says that nonforking occurs. It is similar to simplicity axioms considered in other contexts. From it, we derive the rest of necessary properties.

Heirs and Local Character

Let $p \in gS(N)$. Then p is an heir over $M \prec N$ iff for every small $M^- \prec M$ and small $M^- \prec N^- \prec N$, there is $f : N^- \to_{M^-} M$ so $f(p \upharpoonright N^-) \leq p$.
Theorem: If K does not have the order$_2$ property, then nonforking and heiring are equal. They are always dual.
Theorem: If K has no order$_2$ property and is categorical above κ, then the universal local character is ω.

Large Cardinals

Large cardinals imply the hypotheses [2]

- If κ is strongly compact, then K is short and tame and any $< \kappa$ universal model is an extension base.

Similar results follow from measurable and weakly compact.

Uniqueness of Limit models

Uniqueness of limit models is a proposed dividing line in the classification theory of AECs as a generalization of the union of saturated models. When the universal local character is ω, every limit model is unique over the base.

References

1. W. Boney, R. Grossberg. *Nonforking in short and tame AECs*, Submitted.
2. W. Boney. *Tameness from large cardinal axioms*, Submitted.
3. S. Shelah. *Non-structure theory*, In preparation.