Measure Algebras

Thomas Jech

Warsaw, October 2013

Boolean σ-algebras

Boolean algebra: a set B, operations $a \vee b$, $a \wedge b$ and $-a$; a partial ordering $a \leq b$ and the smallest and the greatest element, $\mathbf{0}$ and $\mathbf{1}$.

If every countable subset A of B has a least upper bound $\bigvee A$ (and the greatest lower bound $\bigwedge A$) then B is a Boolean *σ-algebra*.

B^+ is the set of all nonzero elements of B.
An *antichain* in B is a set $A \subset B^+$ whose any two elements a, b are *disjoint*, i.e. $a \wedge b = \mathbf{0}$. A is *maximal* if $\bigvee A = \mathbf{1}$.

B satisfies the *countable chain condition* (ccc) if it has no uncountable antichains.

Measure algebras

A *measure* on a Boolean σ-algebra B is a real-valued function m with the following properties:

- $m(\mathbf{0}) = 0$, $m(a) > 0$ for $a \neq \mathbf{0}$, and $m(\mathbf{1}) = 1$,
- $m(a) \leq m(b)$ if $a \leq b$,
- $m(a \vee b) = m(a) + m(b)$ if $a \wedge b = \mathbf{0}$,
- if $a_1 > a_2 > a_3 > \ldots$ and $\bigwedge_n a_n = \mathbf{0}$, then $\lim m(a_n) = 0$.

(It follows that $m(\bigvee_{n\in\omega} a_n) = \sum_{n\in\omega} m(a_n)$ whenever the a_n are pairwise disjoint.)

A Boolean σ-algebra that carries a measure is a *measure algebra*.

For every n let $C_n = \{a \in B : m(a) \geq 1/n\}$. We have $C_1 \subset C_2 \subset C_3 \subset ...$ and $\bigcup C_n = B^+$.

If $A \subset C_n$ is an antichain then it has size at most n.
Hence every antichain is at most countable, and so every measure algebra satisfies ccc.

Definition. A Boolean σ-algebra B is *weakly distributive* if for any sequence $\{A_n\}_n$ of maximal antichains there exist finite sets $E_n \subset A_n$ such that
$\bigvee_n \bigwedge_{k \geq n} (\bigvee E_k) = \mathbf{1}$.

Every measure algebra is weakly distributive: Given $\{A_n\}_n$, we let $E_n \subset A_n$ be such that $m(\bigvee E_n) \geq 1 - 1/2^{n+1}$.

The problem of von Neumann.
John von Neumann asked in 1937 if ccc and weak distributivity are sufficient for a Boolean σ-algebra to carry a measure.

Convergence
For any sequence $\{a_n\}_n$ in B let $\limsup a_n = \bigwedge_n \bigvee_{k \geq n} a_k$.

A sequence $\{a_n\}_n$ *converges to* $\mathbf{0}$, $\lim a_n = \mathbf{0}$, if $\limsup a_n = \mathbf{0}$.

- If $a_1 > a_2 > a_3 > \ldots$ and $\bigwedge_n a_n = \mathbf{0}$ then $\lim a_n = \mathbf{0}$.
- If $\{a_n\}_n$ is an antichain then $\lim a_n = \mathbf{0}$.
- If m is a measure and $\lim a_n = \mathbf{0}$, then $\lim m(a_n) = 0$.
- If $\sum_n m(a_n) < \infty$ (e.g., if $m(a_n) \leq 1/2^n$) then $\lim a_n = \mathbf{0}$.

Definition. A Boolean σ-algebra B is *concentrated* if whenever $\{A_n : n = 1, 2, 3, ...\}$ is a sequence of finite antichains with $|A_n| \geq 2^n$ then we can choose $a_n \in A_n$ such that $\lim a_n = \mathbf{0}$.

B is *uniformly concentrated* if there is a choice function F acting on finite antichains so that whenever $\{A_n : n = 1, 2, 3, ...\}$ is a sequence of finite antichains with $|A_n| \geq 2^n$ then, letting $a_n = F(A_n)$, we have $\lim a_n = \mathbf{0}$.

Every measure algebra is uniformly concentrated: Let m be a measure on B. If A is a finite antichain, let $F(A)$ be an element of A of least possible measure. If $|A| \geq 2^n$ then $m(F(A)) \leq 1/2^n$.

Algebraic characterizations of measure algebras

Definition. A Boolean σ-algebra B is *uniformly weakly distributive* if for every n there is a function F_n acting on maximal antichains such that $F_n(A)$ is a finite subset of A, and
for any sequence $\{A_n\}_n$ of maximal antichains, letting
$E_n = F_n(A_n)$, we have
$\bigvee_n \bigwedge_{k \geq n} (\bigvee E_k) = \mathbf{1}$.

Every measure algebra is uniformly weakly distributive: Given $\{A_n\}_n$, we let $E_n \subset A_n$ be such that $m(\bigvee E_n) \geq 1 - 1/2^{n+1}$.

Algebraic characterizations of measure algebras

Theorem 1. A Boolean σ-algebra B is a measure algebra if and only if it is weakly distributive and uniformly concentrated.

Theorem 2. A Boolean σ-algebra B is a measure algebra if and only if it is uniformly weakly distributive and concentrated.

- Every Suslin algebra is ccc, weakly distributive and concentrated.
- Under **PFA,** every ccc, weakly distributive and concentrated Boolean σ-algebra is a measure algebra.

Remark. Under **PFA,** every ccc weakly distributive Boolean σ-algebra is uniformly weakly distributive.

Fragmentations of B

Definition A *fragmentation* of a Boolean σ-algebra B is a chain $C_1 \subset C_2 \subset C_3 \subset \ldots$ with $\bigcup_n C_n = B^+$.
A fragmentation is

(1) *tight* if whenever $a_n \notin C_n$ for all n, then $\lim a_n = \mathbf{0}$,

(2) G_δ if for each n, no sequence in C_n converges to $\mathbf{0}$,

(3) *σ-bounded cc* for every n there is an integer K_n such that every antichain $A \subset C_n$ has size at most K_n.

Remark Condition (2) implies that $\{C_n\}_n$ is *σ-finite cc*, i.e. every antichain in every C_n is finite.

Let m be a measure on B. The *canonical fragmentation* of the measure algebra B is defined by

$$C_n = \{a \in B : m(a) \geq 1/2^n\}.$$

The canonical fragmentation is tight, G_δ and σ-bounded cc.

Theorem 3. A Boolean σ-algebra B is a measure algebra if and only if it has a fragmentation that is tight, G_δ and σ-bounded cc.

Remarks:

- B has a tight σ-bounded cc fragmentation if and only if it is uniformly concentrated.
- B has a tight G_δ fragmentation if and only if it is uniformly weakly distributive.
- The tight G_δ fragmentation is essentially unique: Any two such fragmentations are mutually cofinal, i.e. for each n there is a k such that $C_n \subset C'_k$.

Maharam algebras

A *continuous submeasure* on a Boolean σ-algebra B is a real-valued function m with the following properties:

- $m(\mathbf{0}) = 0$, $m(a) > 0$ for $a \neq \mathbf{0}$, and $m(\mathbf{1}) = 1$,
- $m(a) \leq m(b)$ if $a \leq b$,
- $m(a \vee b) \leq m(a) + m(b)$,
- if $a_1 > a_2 > a_3 > \dots$ and $\bigwedge_n a_n = \mathbf{0}$, then $\lim m(a_n) = 0$.

A Boolean σ-algebra that carries a continuous submeasure is a *Maharam algebra*.

- Every Maharam algebra is uniformly weakly distributive.
- If B is a Maharam algebra then its canonical fragmentation is tight and G_δ.

Maharam algebras

Theorem 4 (Talagrand). There exists a Maharam algebra that is not a measure algebra.

Theorem 5 (Balcar-Jech) A Boolean σ-algebra is a Maharam algebra if and only if it is uniformly weakly distributive.

Theorem 6 (Todorčević) A Boolean σ-algebra is a Maharam algebra if and only if it is weakly distributive and σ-finite cc.

Forcing

The following characterization uses the language of forcing. A complete Boolean algebra B is a measure algebra if and only if Player II has a winning strategy in the following infinite game:

The nth move of Player I is a B-name $\dot{f}(n)$ for an integer and the nth move of Player II is an integer $g(n)$. Thus I produces a B-name $\dot{f}$ for a function from ω to ω and II produces a function $g : \omega \to \omega$. Player II wins if it is forced that

$$\exists N \forall n \geq N \, \dot{f}(n) < g(n) \text{ and } \dot{f}(n) \not\equiv g(n) \bmod 2^n.$$

History

The problem of algebraic characterization of measure algebras is due to von Neumann (1937). The present result uses various earlier work, most notably Maharam (1947), Kelley (1959), Kalton and Roberts (1983), Balcar et al. (1998, 2003), Todorčević (2000, 2004), Talagrand (2007).

Reference

T. Jech: Algebraic characterizations of measure algebras, Proc. AMS 136 (2008), 1285-1294.