The importance of approximate counting in bounded arithmetic

Leszek Kołodziejczyk
University of Warsaw

(based on joint work with Buss-Thapen and Buss-Zdanowski)

Mostowski100
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Bounded arithmetic

Bounded arithmetic: collective name for some first-order arithmetic theories with induction only for (some or all) bounded formulas.

Usual language (mostly due to Sam Buss):
$+, \cdot, \leq, 0, 1, \log x, 2^{\log x \cdot \log y}, \lfloor x/2^y \rfloor$.

Motivation:

- connections to computational complexity,
- connections to propositional proof complexity: arithmetical proofs can be translated into short propositional proofs,
- foundational concerns: how much "finite mathematics" can be done without the exponential function.

Connection to Mostowski (high-level)

Mostowski

did work on

first-order arithmetic

which spawned

bounded arithmetic

which is the topic of

this talk.

Connection to Mostowski (not-so-high-level)

Mostowski

defined the

Kleene-Mostowski hierarchy

which has an important analogue in

bounded arithmetic

which is the topic of

this talk.

Formula and theory hierarchies

Analogue of Kleene-Mostowski hierarchy:

$\hat{\Sigma}^b_n$ formulas: $\exists x_1 < t_1 \forall x_2 < t_2 \ldots Q x_n < t_n \, \psi$,
where ψ sharply bounded (only quantifiers of the form $Qx < \log t$).

Full BA: basic axioms + induction for all bounded formulas.
The fragment T^n_2: induction only for $\hat{\Sigma}^b_n$ formulas.
(Definition and strength of T^0_2 very sensitive to choice of language.)

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Expressive power:

- $\hat{\Sigma}^b_n \leftrightsquigarrow \Sigma^p_n$, the n-th level of the polynomial time hierarchy.
  So, for instance, $\hat{\Sigma}^b_1 \leftrightsquigarrow$ = NP.
- Provably $\hat{\Delta}^b_n$ in $T^n_2 \leftrightsquigarrow$ polynomial time with Σ^b_{n-1} oracle.
  (Where $\hat{\Delta}^b_n$ means definable by both $\hat{\Sigma}^b_n$ and negated $\hat{\Sigma}^b_n$ flas.)

Witnessing theorems

The connection with computational complexity runs deeper, in the form of witnessing theorems. For example:

- If $T^0_2 \vdash \forall x\, \exists y\, \psi(x, y)$ with $\psi \in \hat{\Sigma}^b_0$, then given x as input, y can be found in polynomial time.
- If $T^1_2 \vdash \forall x\, \exists y\, \psi(x, y)$ with $\psi \in \hat{\Sigma}^b_0$, then given x as input, y can be found by a *polynomial local search* procedure.

Fundamental problem:

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“Is (full) BA finitely axiomatizable?”,

or equivalently,

“Is BA equivalent to one of the theories T^n_2?”

or equivalently,

“Does BA prove the collapse of the polynomial-time hierarchy?”

(No matter how you state it, the question is apparently out of reach.)

The relativized setting

To get some unprovability results, it helps to consider relativized BA, with a new "oracle" predicate α (which leads to $\hat{\Sigma}^b_n(\alpha)$, $T^n_2(\alpha)$ etc.).

For instance:

- $T^0_2(\alpha) \subsetneq T^1_2(\alpha) \subsetneq T^2_2(\alpha) \ldots$
  (Krajíček-Pudlák-Takeuti 1991),
- $\mathrm{BA}(\alpha) \nvdash \mathrm{PHP}(\alpha)$,
  viz. that for all a, α is not an injective function from $a + 1$ to a
  (strengthening of Ajtai 1988, due to Beame et al. 1992).

Two problems from the research frontier

1. Can the theories $T^n_2(\alpha)$ be separated by a $\forall\hat{\Sigma}^b_1(\alpha)$ sentence?
   - only $\mathrm{T}^0_2(\alpha) \not\preccurlyeq_{\forall\hat{\Sigma}^b_1(\alpha)} T^1_2(\alpha) \not\preccurlyeq_{\forall\hat{\Sigma}^b_1(\alpha)} T^2_2(\alpha)$ known.
2. An "interesting" independence result for $\mathrm{BA}(\alpha)$ with a parity quantifier, "there is an odd number of $x < t$ such that".
   - e.g. for $\mathrm{PHP}(\alpha)$, or maybe for the counting principle mod 3: there is no partition of $\{0, \ldots, 3n\}$ into 3-element sets.

Both problems have very closely corresponding versions in propositional proof complexity.

The weak pigeonhole principle

iWPHP($\mathcal{F}$) says:
no function $f \in \mathcal{F}$ is an injection from $2a$ into a (where $a > 0$).

Paris, Wilkie and Woods 1988:

- $\mathrm{BA}(\alpha) \vdash \mathrm{iWPHP}(\alpha)$,
- BA $\vdash$ if there are no primes between a and a^{11}, then there is a bounded-definable injection from $9a \log a$ into $8a \log a$. (So, in particular, there *are* primes between a and a^{11}.)

WPHP in the bounded arithmetic hierarchy

Theorem (essentially Maciel-Pitassi-Woods 2002)
$T^2_2(\alpha) \vdash \mathrm{iWPHP}(\alpha)$.

Theorem (Chiari-Krajíček 1998)
$T^1_2(\alpha) \nvdash \mathrm{iWPHP}(\alpha)$.

The surjective WPHP and approximate counting

sWPHP($\mathcal{F}$) says:
no function $f \in \mathcal{F}$ is a surjection from a onto $2a$ (where $a > 0$).

Theorem (Jeřábek 2009)

The theory $T^1_2 + \mathrm{sWPHP}(\hat{\Delta}^b_2)$ can perform approximate counting of $\hat{\Sigma}^b_1$-definable sets: given bounded and $\hat{\Sigma}^b_1$-definable X it finds surjections witnessing $s \twoheadleftarrow X \twoheadleftarrow s + s/\mathrm{polylog}(s)$ for some s.

- J. actually needs to prohibit surjections from a onto $a+a/\log(a)$, but this is to a large extent conservative over sWPHP.
- A weak form of aprx. counting is available in $T^0_2 + \mathrm{sWPHP}(\hat{\Delta}^b_1)$. The two theories are sometimes called APC_1 and APC_2.

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$\forall\hat{\Sigma}^b_1(\alpha)$ principles separating low levels of the hierarchy

- iWPHP(α), with "x maps to y" formalized not by $\alpha(x, y)$, but by: $\forall i < \log a\, [\alpha(x, i) \equiv (\mathrm{bit}(y, i) = 1)]$.
- Ramsey's principle: the graph determined by α on $[0, b)$ has a homogeneous set of size $(\log b)/2$.
- Herbrandized ordering principle HOP: if $\preceq\restriction_{[0,c)}$ is a linear ordering, then h cannot be the associated *total* predecessor function. (Here $\preceq$ and h given by the oracle α.)

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All three of these (and more) are provable in $\mathrm{APC}_2(\alpha)$!

Status of HOP

Proposition

Both $T^2_2(\alpha)$ and $\mathrm{APC}_2(\alpha)$ prove HOP.

Proof.

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APC_2 is known to prove the tournament principle:
given a tournament, there is a log-sized dominating set.
Apply this to the tournament given by $\preceq$.
Finding the least element of the log-sized set can be done in T^0_2. □

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Independence for $T^1_2(\alpha) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\alpha))$

Theorem

$T^1_2(\alpha) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\alpha)) \nvdash \mathrm{HOP}$.

Independence for $T^1_2(\alpha) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\alpha))$

Theorem

$T^1_2(\alpha) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\alpha)) \nvdash \mathrm{HOP}$.

Proof sketch

Assume $T^1_2(\preceq, h) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\preceq, h)) \vdash \mathrm{HOP}$. Then there exist:

- a polytime $f^{\preceq,h}(\cdot,\cdot)$ such that $f(a,\cdot) : [0,a^2) \to [0,a)$ for each a,
- a *polynomial local search* procedure with oracles $\preceq, h, r_1, r_2$ that takes input c and finds either a witness to $\mathrm{HOP}\restriction_{[0,c)}$ or some $a \in [c, t(c))$ such that $r_1(a) = r_2(a)$ or $f(a, r_1(a)) \neq f(a, r_2(a))$.

Let's pretend that the PLS procedure is just a polytime function g, which runs for $\mathrm{polylog}(c)$ steps and asks queries: "$x_1 \preceq x_2$?", "$h(x) = ?$", "$r_1(y) = ?, r_2(y) = ?$"

$T^1_2(\alpha) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\alpha))$: fooling g

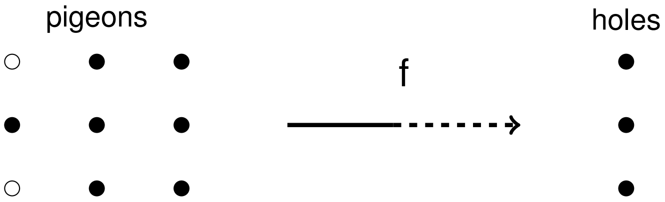
We gradually define $\preceq\restriction_{[0,c)}$ so as to answer the queries without revealing a witness to HOP. For queries about $\preceq$ and h, this is easy. We define $\preceq$ on one or two more points.

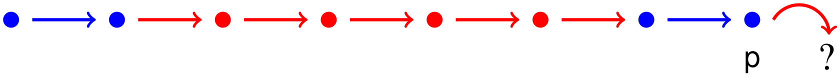
$T^1_2(\alpha) + \text{iWPHP}(\hat{\Delta}^b_1(\alpha))$: fooling g

For queries about r_1 and r_2, we have to extend $\preceq$ by $\text{polylog}(c)$ more points so that $r_1(a)$ and $r_2(a)$ give a collision in $f(a, \cdot)$.

$T^1_2(\alpha) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\alpha))$: fooling g

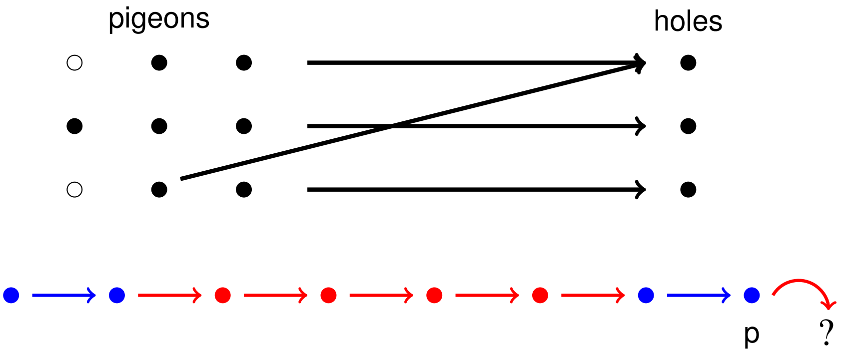
We do this in stages. At each stage, we think of $\preceq$ as defined on all of $[0, c)$, but only part is settled; the rest is tentative. Also, at each stage $\gg a$ of the a^2 pigeons are still active; the rest have been *discarded*.





$T^1_2(\alpha) + \mathrm{iWPHP}(\hat{\Delta}^b_1(\alpha))$: fooling g

At a stage, if $> a$ active pigeons make it through the computation of $f(a,\cdot)$ without querying h of the currently $\preceq$-smallest point p, there is a collision in $f(a,\cdot)$ that we can use to define $r_1(a), r_2(a)$.



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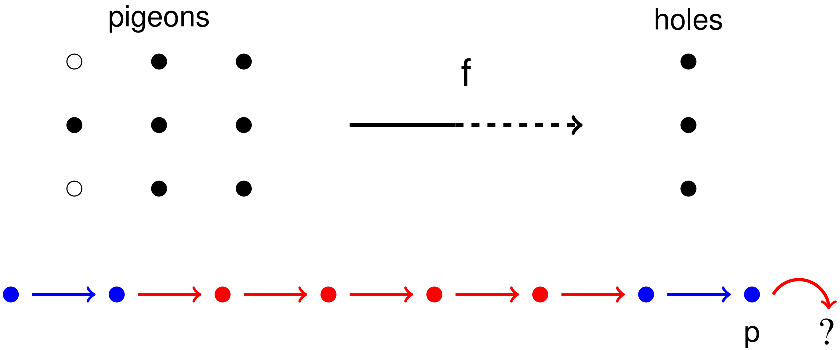
Otherwise, we find a tentative point q which is queried by few of the active pigeons and move it below p.
Discard the pigeons which do not query p or do query q.



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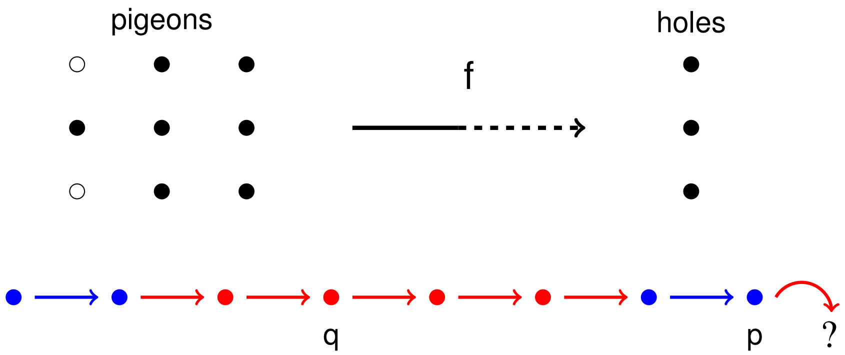

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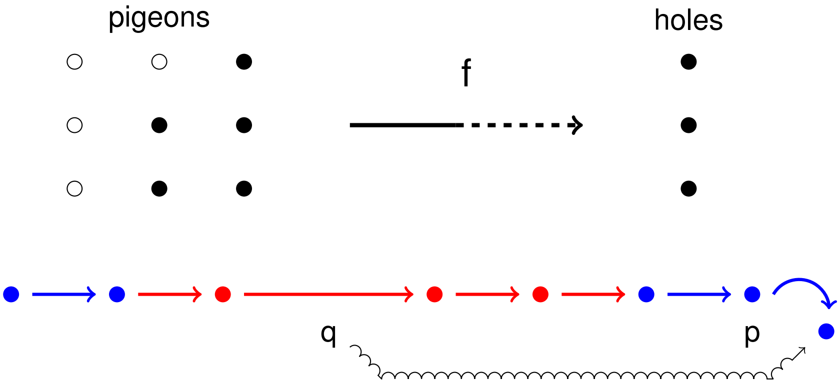

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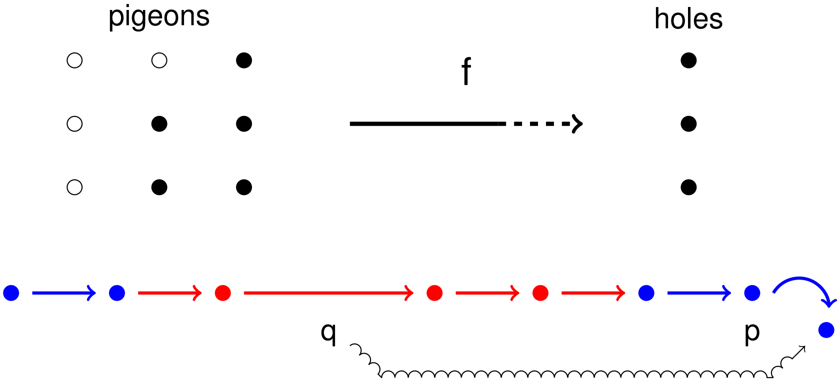
In this way, all remaining active pigeons make it one step further in the computation of $f(a,\cdot)$ without querying the $\preceq$-least point.
After polylog(c) many stages, we find a “safe” collision in $f(a,\cdot)$.



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Theories with the $\oplus$ quantifier

$\oplus x < y$:= "there is an odd number of $x < y$ such that".

$\mathrm{BA}^{\oplus}$: induction for bounded formulas in the language with $\exists, \forall, \oplus$.

$\hat{\Sigma}_n^{b,\oplus\mathrm{P}}$ formulas: $\exists x_1 < t_1 \forall x_2 < t_2 \dots Q x_n < t_n\, \psi$,
where ψ open except for perhaps $\oplus$ in front of Σ_0^b formulas.

$T_2^{n,\oplus\mathrm{P}}$: induction for $\hat{\Sigma}_n^{b,\oplus\mathrm{P}}$. Note that $\bigcup_n T_2^{n,\oplus\mathrm{P}} \neq \mathrm{BA}^{\oplus}$.

This all relativizes smoothly to α.

Collapse for theories with $\oplus$

Theorem

$\mathrm{BA}^{\oplus}$ *is conservative over* $\mathrm{APC}_2^{\oplus \mathrm{P}}$*, also in the relativized setting.*

Proof sketch:

- The idea is to formalize *Toda's Theorem*: a known collapse result for bounded formulas involving $\exists, \forall, \oplus$.
- We inductively assign to each bounded formula $\varphi(x)$ with $\oplus$ a $\hat{\Sigma}_0^b$ formula $\psi(x, y, r)$ such that

$$\varphi(x) \Rightarrow \text{ w.h.p. over } r, \ \oplus y < t \, \psi(x, y, r),$$

$$\neg\varphi(x) \Rightarrow \text{ w.h.p. over } r, \ \neg \oplus y < t \, \psi(x, y, r).$$

- On a bounded interval, "w.h.p. over r, $\oplus y < t \, \psi(x, y, r)$" is $\hat{\Delta}_1^{b, \oplus \mathrm{P}}$, and already $T_2^{0, \oplus \mathrm{P}}$ has induction for it. □

The collapse result: comments on proof

- A crucial step in the induction is dealing with $\exists$, based on the so-called Valiant-Vazirani Lemma: given $\hat{\Sigma}^b_1$ formula $\varphi(x)$, there is a $\hat{\Sigma}^b_0$ formula $\psi(x, y, r)$ such that

$$\varphi(x) \Rightarrow \Pr_r[\exists! y < t\, \psi(x, y, r)] > 1/t(x) \text{ for some term } t,$$
$$\neg\varphi(x) \Rightarrow \Pr_r[\exists y < t\, \psi(x, y, r)] = 0.$$

- To get this, we need to know things like: given a propositional formula in n variables, there is $k \leq n$ such that the formula has between 2^{k-2} and 2^k satisfying assignments.
  This seems to engage the full power of the approx. counting.
- The other inductive steps are more or less natural, but we must make sure that correctness of the translation can be verified in $\mathrm{APC}_2^{\oplus \mathrm{P}}$, particularly in the case of nested $\oplus$'s.

The collapse result: propositional consequences

Theorem

Any simple enough (say, DNF) formula which has a proof of size s in the system with connectives $\wedge, \vee, \neg, \oplus$ and formulas of depth $\leq d$, has a proof of size at most $s^{\log^{c(d)}(s)}$ with formulas of depth ≤ 3.

Proof sketch:

- $\mathrm{BA}^{\oplus}(\alpha)$ proves a reflection principle for the depth d system.
- So, $\mathrm{APC}_2^{\oplus\mathrm{P}}(\alpha)$ and hence $T_2^{3,\oplus\mathrm{P}}(\alpha)$ proves it too.
- Proofs in $T_2^{3,\oplus\mathrm{P}}(\alpha)$ translate into short proofs in the depth 3 system (Paris-Wilkie translation from arithm. to prop. logic).
- So, the depth 3 system proves reflection for the depth d system.
- The simulation follows. □

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Maybe the model-theoretic properties of WPHP could help?