Small Polish structures

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Polish structures

Definition (K.)

A Polish structure is a pair (X, G), where G is a Polish group acting faithfully on a set X so that $G_x <_c G$ for every $x \in X$.

Examples

1. Compact [profinite] structures: X - a compact metric [profinite] space, G - a compact group, the action continuous,
2. [Polish] G-spaces: X - a [Polish] space, G - a Polish group, the action is continuous, e.g.:
   - X - a compact metric space, $G = \mathrm{Homeo}(X)$ - the group of all homeomorphisms of X with the compact-open topology,
   - X - a compact metric group, $G = \mathrm{Aut}(X)$ - the group of all topological automorphisms of X with the c-o topology,
3. Borel G-spaces: X - a Polish space, G - a Polish group, the action is Borel-measurable.

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3. Borel G-spaces: X - a Polish space, G - a Polish group, the action is Borel-measurable.

Smallness

Definition (K.)

A Polish structure (X, G) is small if for every $n \in \omega$ there are only countably many orbits on X^n.

Examples of small Polish structures (K.)

1. $(S^n, \mathrm{Homeo}(S^n))$, $n \in \omega$,
2. $(I^n, \mathrm{Homeo}(I^n))$, $n \in \omega \cup \{\omega\}$,
3. $((S^1)^n, \mathrm{Homeo}((S^1)^n))$, $n \in \omega \cup \{\omega\}$,
4. $(P, \mathrm{Homeo}(P))$, P - the pseudo-arc,
5. $(H, \mathrm{Aut}(H))$, H - a profinite abelian group of finite exponent, $\mathrm{Aut}(H)$ - the group of all topological automorphisms of H,
6. $(H, \mathrm{Aut}^0(H))$, H - as above, $\mathrm{Aut}^0(H)$ - the group of all automorphisms of H preserving a distinguished inverse system indexed by ω.

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Polish topology on X

G - a Polish group acting on a set X
$\mathcal{U} := \{U \cdot x : U \subseteq_o G, x \in X\}$

Proposition (Dobrowolski)

1. The family $\mathcal{U}$ is a basis of a topology on X. Denote this topology by τ.
2. The action of G on (X, τ) is continuous.
3. $(\forall x \in X)(G_x \leq_c G) \iff \tau$ is T_1.
4. X is a disjoint union of G-orbits which are all clopen in τ-topology; moreover, for every $x \in X$, $G/G_x \approx G \cdot x$ (where the orbit $G \cdot x$ is equipped with topology τ).

Corollary (Dobrowolski)

If (X, G) is a small Polish structure, then (X, τ) is Polish, and so (X, G) is a Polish G-space with X equipped with topology τ.

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m-independence

(X, G) - a compact [profinite] structure
a - finite tuple of elements of X
A, B - finite subsets of X
$o(a/A) := \{g \cdot a : g \in G_A\}$

Definition (Newelski)

$a \overset{m}{\perp}_A B \iff o(a/AB) \subseteq_o o(a/A)$,
$a \overset{m}{\not\perp}_A B \iff o(a/AB) \subseteq_{nwd} o(a/A)$.

Fact (Newelski)

$\overset{m}{\perp}$ is invariant, symmetric and transitive. If (X, G) is small, then $\overset{m}{\perp}$ satisfies the existence of independent extensions.

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$\pi_A : G_A \to o(a/A)$ is defined by $\pi_A(g) = g \cdot a$.

Definition (K.)

$a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B \iff \pi_A^{-1}[o(a/AB)] \subseteq_{nm} \pi_A^{-1}[o(a/A)]$,
$a \not\mathop{\smile\!\!\!\!\!\!|}^{m}_A B \iff \pi_A^{-1}[o(a/AB)] \subseteq_m \pi_A^{-1}[o(a/A)]$.

Remark (K.)

$a \mathop{\smile\!\!\!\!\!\!|}^{nm}_A B \iff G_{AB} G_{Aa} \subseteq_{nm} G_A$,

Theorem (K.)

$\mathop{\smile\!\!\!\!\!\!|}^{nm}$ is invariant, symmetric and transitive. If (X, G) is small, then $\mathop{\smile\!\!\!\!\!\!|}^{nm}$ satisfies the existence of independent extensions.

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Properties of *nm*-independence in details

(X, G) - a Polish structure

For a finite $A \subseteq X$, define
$\mathrm{Acl}(A) := \{x \in X^n : n \in \omega \setminus \{0\}$ and $o(x/A)$ is countable$\}$.

1. (Invariance) $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B \iff g(a) \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_{g[A]} g[B]$ whenever $g \in G$ and $a, A, B \subseteq X$ are finite.
2. (Symmetry) $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_C b \iff b \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_C a$ for every finite $a, b, C \subseteq X$.
3. (Transitivity) $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_B C \wedge a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B \iff a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A C$ for every finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$.
4. For every finite $a, A \subseteq X$, $a \in \mathrm{Acl}(A)$ iff for all finite $B \subseteq X$ we have $a \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$.
5. (Existence of independent extensions) Assume (X, G) is small. Then, for all finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there is $b \in o(a/A)$ such that $b \mathop{\smile\!\!\!\!\!\!\;|}^{nm}_A B$.

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nm-independence in a topological context

Proposition (Dobrowolski)

Let (X, G) be a Polish structure and let $a, A, B \subseteq X$ be finite. Then $a \mathop{\smile\!\!\!\!\!\!\downarrow}^{m}_{A} B \iff o(a/B) \subseteq_{nm} o(a/A)$, where X is equipped with topology τ_A induced by the action of G_A on X.

(X, G) - a Polish structure, where G acts continuously on a Hausdorff space X
Let $a, A, B \subseteq X$ be finite.

Proposition (K.)

If $o(a/A)$ is non-meager in itself, then $a \mathop{\smile\!\!\!\!\!\!\downarrow}^{m}_{A} B \iff o(a/AB) \subseteq_{nm} o(a/A)$.

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In compact [profinite] structures, $\mathop{\smile\!\!\!\!\!\!\downarrow}^{m} = {}^{m}\!\mathop{\smile\!\!\!\!\!\!\downarrow}$.

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$\mathcal{NM}$-rank and *nm*-stability

(X, G) - a small Polish structure

Definition (K.)

$\mathcal{NM}$: orbits over finite sets $\rightarrow Ord \cup \{\infty\}$

$\mathcal{NM}(a/A) \geq \alpha + 1$ iff there is a finite set $B \supseteq A$ such that $a \not\downarrow^{nm}_A B$ and $\mathcal{NM}(a/B) \geq \alpha$.

Definition (K.)

$\mathcal{NM}(X) = \sup\{\mathcal{NM}(x/\emptyset) : x \in X\}$.

Definition (K.)

(X, G) is *nm*-stable if for every $x \in X$, $\mathcal{NM}(x/\emptyset) < \infty$.

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$\mathcal{NM}$-rank – cont.

Remark (K.)

$\mathcal{NM}(a/A) = 0$ iff $o(a/A)$ is countable.

Lascar Inequalities

$\mathcal{NM}(a/bA) + \mathcal{NM}(b/A) \leq \mathcal{NM}(ab/A) \leq \mathcal{NM}(a/bA) \oplus \mathcal{NM}(b/A)$.

Definition (K.)

Let (X, G) be a small G-space and $D \subseteq X$. We say that D is $*$-closed if it is closed and invariant over some finite $A \subseteq X$ (i.e. invariant under G_A). For such a D, $\mathcal{NM}(D) := \sup\{\mathcal{NM}(d/A) : d \in D\}$.

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Examples

1. In $(S^n, \mathrm{Homeo}(S^n))$, $\mathcal{NM}(S^n) = 1$.
2. In $(I^\omega, \mathrm{Homeo}(I^\omega))$, $\mathcal{NM}(I^\omega) = 1$.
3. $(P, \mathrm{Homeo}(P))$ where P is the pseudo-arc is not *nm*-stable.
4. In $((\mathbb{Z}_{p^n})^\omega, \mathrm{Aut}((\mathbb{Z}_{p^n})^\omega))$, $\mathcal{NM}((\mathbb{Z}_{p^n})^\omega) = n$.

Remark (K.)

Everything that was said about $\mathop{\smile\!\!\!\!\!\mid}^{nm}$ works in a suitably defined imaginary extension X^{eq} of X (e.g. if G acts continuously on a space X and E is an invariant, closed equivalence relation on X^n, then $X^n/E \subseteq X^{eq}$).

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1. A Polish group structure is a Polish structure (H, G) such that H is a group and G acts as a group of automorphisms of H.
2. A (topological) G-group is a Polish group structure (H, G) such that H is a topological group and the action of G on H is continuous.
3. A Polish [compact] G-group is a topological G-group (H, G), where H is a Polish [compact] group.

Main examples of compact G-groups

1. $(H, \mathrm{Aut}(H))$, where H is a compact metric group.
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nm-generics

(H, G) - a small Polish group structure

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We say that the orbit $o(a/A)$ is *nm*-generic (or that a is *nm*-generic over A) if for all $b \in H$ with $a \mathop{\smile\!\!\!\!\!\!\,|}^{nm}_A b$ one has that $b \cdot a \mathop{\smile\!\!\!\!\!\!\,|}^{nm} A, b$.

Proposition (K.)

Let (H, G) be a small Polish G-group. Then for every finite $A \subseteq H$ and $a \in H$, the orbit $o(a/A)$ is *nm*-generic iff $o(a/A) \subseteq_{nm} H$. In particular, an *nm*-generic orbit (over any given finite $A \subseteq H$) exists.

Proposition (Dobrowolski [see his poster])

There is a small Polish group structure (H, G) without *nm*-generic orbits. Hence, there is no Polish topology on H making (H, G) a Polish G-group.

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$\mathcal{NM}$-rank in small Polish G-groups

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Assume (H, G) is *nm*-stable. Then, an element $a \in H$ is generic over a given finite set $A \subseteq H$ iff $\mathcal{NM}(a/A) = \mathcal{NM}(H)$.

$H_2 \leq H_1$ - $*$-closed subgroups of H

Lascar Inequalities for groups

$\mathcal{NM}(H_2) + \mathcal{NM}(H_1/H_2) \leq \mathcal{NM}(H_1) \leq \mathcal{NM}(H_2) \oplus \mathcal{NM}(H_1/H_2)$.

Corollary (K.)

Suppose $\mathcal{NM}(H_1) < \infty$. Then

$$\mathcal{NM}(H_1) = \mathcal{NM}(H_2) \iff H_2 \leq_o H_1.$$

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Suppose $\mathcal{NM}(H_1) < \infty$. Then

$$\mathcal{NM}(H_1) = \mathcal{NM}(H_2) \iff H_2 \leq_o H_1.$$

General goal

To understand the structure (or even classify) small compact [or, more generally, Polish] G-groups. In other words, we would like to undestand the algebraic/topological consequences of the assumption of smallness in the context of compact [or Polish] G-groups.

Small Polish G-groups

Proposition (K.)

1. If (H, G) is a small Polish G-group, and $S \subseteq H$ is finite, then $\langle S \rangle$ is countable, i.e. $\langle S \rangle$ does not have limit points.
2. If (H, G) is a small compact G-group, then H is locally finite, and so H is a profinite (so 0-dimensional) group.

Example (K.)

Consider the additive structure on $\mathbb{Q}$. Take the discrete topology on $\mathbb{Q}$ and the product topology on $\mathbb{Q}^\omega$. For a suitably chosen Polish group G, $(\mathbb{Q}^\omega, G)$ is a small Polish G-group of $\mathcal{NM}$-rank 1, which is torsion-free (so not locally finite) and 0-dimensional.

Proposition (Dobrowolski [see his poster])

There is a 1-dimensional small Polish G-group.

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Conjectures on small profinite groups

Remark (K.)

If (H, G) is a small compact G-group, then H is a profinite group. But G is not necessarily compact, so (H, G) needn't be a small profinite group regarded as profinite structure.

(H, G) - a small profinite group regarded as profinite structure

Newelski's Conjecture

H is abelian-by-finite.

Intermediate Conjectures

(A) H is solvable-by-finite.

(B) If H is solvable-by-finite, then it is nilpotent-by-finite.

(C) If H is nilpotent-by-finite, then it is abelian-by-finite.

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Results on small profinite groups

Theorem (Wagner)

Each small, nm-stable profinite group is abelian-by-finite.

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The conjectures for small compact G-groups

From now on, we consider generalizations of Conjectures (A), (B), (C) to the wider context of small compact G-groups. It turns out that in general they are all false.

Counter-example to (A)

H - any finite non-solvable group
S_∞ - the group of all permutations of $\mathbb{N}$
S_∞ acts on H^ω by $\sigma\langle h_0, h_1, \dots\rangle = \langle h_{\sigma(0)}, h_{\sigma(1)}, \dots\rangle$.

Then (H^ω, S_∞) is a small compact S_∞-group, and H^ω is not solvable-by-finite.

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Results on small, nm-stable compact G-groups

(H, G) - a small compact G-group

Theorem (K.)

If (H, G) is nm-stable, then H is nilpotent-by-finite.

Conjecture

If (H, G) is nm-stable, then H is abelian-by-finite.

Theorem (K., Wagner)

If $\mathcal{NM}(H) \leq \omega$, then H is abelian-by-finite.

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Infinite ordinal $\mathcal{NM}$-ranks are possible

Example (K.)

- Choose $\omega = I_0 \supseteq I_1 \supseteq \dots$ so that $I_i \setminus I_{i+1}$ are all infinite and $\bigcap I_i = \emptyset$,
- $H_i := \{\eta \in \mathbb{Z}_p^\omega : \eta(j) = 0 \text{ for all } j \in I_i\}$, $i \in \omega$,
- $G := \{g \in \mathrm{Aut}(\mathbb{Z}_p^\omega) : g[H_i] = H_i \text{ for every } i \in \omega\}$.

Then:

1. $(\mathbb{Z}_p^\omega, G)$ is a small compact G-group,
2. $\mathcal{NM}(\mathbb{Z}_p^\omega) = \omega$,
3. $H_0 <_{nwd} H_1 <_{nwd} \dots$ is an infinite increasing sequence of $\emptyset$-closed subgroups.

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Without *nm*-stability

Question

What can be said about the structure of small compact G-groups without assuming *nm*-stability?

Open questions on small Polish G-groups

Question

Is every small, *nm*-stable Polish G-group abelian-by-countable?

Even the following question is open.

Question

Is every small Polish G-group of $\mathcal{NM}$-rank 1 abelian-by-countable?

Small compact G-rings

Definition

Let G be a Polish group. A compact G-ring is a Polish structure (R, G), where R is a compact topological ring and G acts on R continuously as a group of automorphisms.

Goal

To understand the structure of small compact G-rings. Investigate connections with small compact G-groups.

Proposition (K., Dobrowolski)

If (R, G) is a small compact G-ring, then R is locally finite and profinite.

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A few classical definitions from ring theory

R - a ring

1. An element $r \in R$ is nilpotent of nilexpenent n if $r^n = 0$ and n is the smallest number with this property.
2. R is nil [of nilexponent n] if every element of R is nilpotent [of nilexponent $\leq n$ and there is an element of nilexponent n].
3. R is nilpotent of class n if $r_1 \cdot \ldots \cdot r_n = 0$ for all $r_1, \ldots, r_n \in R$ and n is the smallest number with this property.
   R is nilpotent if it is nilpotent of class n for some n.
4. R is null if $r_1 \cdot r_2 = 0$ for all $r_1, r_2 \in R$.

Results on small, *nm*-stable compact *G*-rings

In various contexts, I have proved certain theorems which allow to deduce some properties of rings from the appropriate properties of groups. As a conclusion of these results and the theorems on small, *nm*-stable compact *G*-groups, we get the following corollary.

Theorem (K.)

1. A small, *nm*-stable compact *G*-ring is nilpotent-by-finite.
2. A small, compact *G*-ring of finite $\mathcal{NM}$-rank is null-by-finite.

Conjecture

A small, *nm*-stable compact *G*-ring is null-by-finite.

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This conjecture is equivalent to the conjecture that each small, *nm*-stable compact *G*-group is abelian-by-finite.

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The structure of small compact G-rings

Theorem (K., Dobrowolski)

Every locally finite profinite ring is (nil of finite nilexponent)-by-(product of complete matrix rings over finite fields with only finitely many factors up to isomorphism). More precisely:

1. The Jacobson radical of a locally finite profinite ring is nil of finite nil exponent.
2. Let R be a topological ring. Then R is semisimple, locally finite profinite ring iff R is a product of complete matrix rings over finite fields with only finitely many factors up to isomorphism.

Question

Can one deduce anything on the structure of small compact G-groups from the above theorem?

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Future investigations of Polish structures

- Prove counterparts of some deep model theoretic results, e.g. a variant of the group configuration theorem.
- Investigate further the structure of groups and rings in the context of small Polish structures.
- Try to apply the introduced model theoretic tools (e.g. $\mathop{\smile\!\!\!\!\!\!|}^{nm}$, $\mathcal{NM}$-rank, *nm*-generics) to some descriptive set theoretic problems.