Model theory of the adeles

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Adeles

K a global field of characteristic 0 (so $[K:\mathbb{Q}] < \omega$). K is very complicated metamathematically, undecidable, and with definable sets arbitrarily high in the arithmetic hierarchy..
To K we attach $\mathbb{A}_K$, the **adeles of K**, a locally compact commutative ring with 1, introduced by A.Weil.

$\mathbb{A}_K$ is a *restricted product* (in a sense to be described below) of the family of all completions $\{K_p\}$ of K at prime divisors p . These completions are individually,and even uniformly, well-behaved metamathematically, being decidable,and admitting quantifier eliminations linked to their topologies.If K is Q, the completions are exactly the real and p-adic fields.For general K one needs also finite extensions of the p-adics,and the complex field. In number theory a major concern is the passage from local to global, i.e from knowing solvability of an equation modulo all powers of primes(including the archimedean ones) to knowing solvability in the integers.

Analysis

$\mathbb{A}_K$ is a global object, constructed out of the various completions,and its analytic structure is very basic in modern number theory.The importance is directly connected to reciprocity laws (for example, quadratic reciprocity) in arithmetic, or, in other words, to hidden uniformities about primes. It is worth recalling that Skolem used local analytic methods in the study of integer points on curves.

Basic Notation

[see Cassels-Frohlich (Tate thesis) for this convenient notation]

- K_p may be $\mathbb{R}$: $|\cdot|_p$ usual absolute value
- K_p may be $\mathbb{C}$: $|\cdot|_p$ square of usual absolute value
- K_p may be p-adic: $|x| = (Np)^{-\nu_p(x)}$ where Np=cardinal of residue field of ν_p

Unit ball

$$\mathcal{O}_p = \{x \in K_p : |x|_p \leq 1\}, \quad \text{compact}$$

Write $\mathcal{P}$ for maximal ideal in the p-adic (nonarchimedean) case

Restricted product

$\mathbb{A}_K$ is a subring of $\prod_p K_p$, consisting of the f such that $\{p : f(p) \notin \mathcal{O}_p\}$ is finite.

$K \to \mathbb{A}_K$ *via*
$\alpha \to$ constant function α

Topology
The K_p have the standard locally compact metric topologies.
$\mathbb{A}_K$ has as a basis of open sets the products $\prod_p U_p$, where U_p is open and equal to $\mathcal{O}_p$ for all but finitely many p

Measure
K_p has Haar measure μ_p normalised so $\mu_p(\mathcal{O}_p) = 1$.
μ_K (or μ if K understood) is Haar measure with $\mu_K(\prod \mathcal{O}_p) = 1$.

The definable sets

We consider sets definable in the ring language, either in $\mathbb{A}_K$ for fixed K, or in $\mathbb{A}_K$ for varying K. Unless otherwise mentioned, "definable" means definable without parameters. For such sets we consider

- their topological structure
- measurability
- measure

via **quantifier elimination**.

Method: lift from the K_p by method of Feferman-Vaught internalized using Boolean algebra of idempotents of $\mathbb{A}_K$.

Boolean algebra $\mathbb{B}_K$

$$\begin{aligned} \mathbb{B}_K &= \{e \in \mathbb{A}_K : \ e^2 = e\} \\ e \wedge f &= ef \\ \neg e &= 1 - e \\ e \vee f &= e + f - ef \end{aligned}$$

Internalized Boolean-valued model

Let P be the set of all p.
Then the set of idempotents of $\mathbb{A}_K$ corresponds to $Powerset(P)$, even as boolean algebras, via

$$e \rightarrow \{p: \quad e(p) = 1\}$$

In Feferman-Vaught theory one considers, for ring formulas $\Phi(\nu_1, ..., \nu_n)$ and $f_1, ..., f_n \in \mathbb{A}_K$

$$(\#) \qquad [[\Phi]]\,(f_1, ..., f_n) = \{p : K_p \models \Phi(f_1(p), .., f_n(p))\} \in powerset(P)$$

and this naturally corresponds to an idempotent

Essential point 1

For fixed Φ, the map

$$\mathbb{A}_K \rightarrow \textit{idempotents}$$

given by ($\#$) is definable in the ring language (even uniformly in K)

A basic ingredient is the correspondence

$$p \rightarrow e_p, \quad e_p(p) = 1, \quad e_p(q) = 0 \textit{ for } q \neq p$$

from P to **minimal idempotents**

Essential point 2

The map $\mathbb{A}_K \to \mathbb{A}_K$ given by

$$x \to e_p \cdot x$$

has kernel $(1 - e_p)\mathbb{A}_K$, and image

$$\begin{array}{ccc} e_p \cdot \mathbb{A}_K & \cong & K_p \\ e_p \cdot x & \leftarrow & x \end{array}$$

Both points are not specific to the use of the K_p, but the next is.

Fact: Uniformly in K and for all p which are not complex, there is a ring-theoretic definition of $\mathcal{O}_p$ (*topology uniformly definable*)

Consequence: uniformly in K one can first-order define the finite idempotents, i.e. those e which are the union of finitely many minimal idempotents (call this set **FIN**)

Feferman-Vaught, first version

Boolean formalism on $\mathbb{B}_K$

Usual $\wedge$, $\vee$, $\neg$, 0, 1 predicates

- $card(e) \leq n$, meaning e has $\leq n$ atoms below it
- $FIN(e)$, meaning e is a finite idempotent

Fact (1950's - Tarski and /or Vaught ?)
$\mathbb{B}_K$ has Q.E. in above formalism

Recall, for $\Phi(\nu_1, ..., \nu_n)$ a ring formula, the map $[[\Phi]] : \mathbb{A}_K^n \to \mathbb{B}_K$

Theorem

For every ring formula $\Phi(\nu_1, ..., \nu_n)$ there are (effectively) ring formulas $\Phi_1(\bar{\nu}), ..., \Phi_r(\bar{\nu})$ and a $\Psi(w_1, ..., w_r)$ from Boolean formalism so that for all K

$$\mathbb{A}_K \models \Phi(\bar{\nu}) \iff B_k \models \Psi([[\Phi_1]](\bar{\nu}), ..., [[\Phi_r]](\bar{\nu}))$$

Feferman-Vaught, continued

To be useful in applications we need to get $\Phi_1, ..., \Phi_r$ of a simple form, and this requires **quantifier elimination** for the K_p.

This we have for fixed K using work of various authors. An essential role is played by solvability predicates $SOL_n(x_1, ..., x_n)$ expressing (in the K_p) that $x_1, ..., x_n \in \mathcal{O}_p$ and

$$y^n + x_1 y^{n-1} + .. + x_n \quad \textit{is solvable in } \mathcal{O}_p/p.$$

[This in turn relates to Riemann hypothesis for curves and ultimately to motivic issues]

Consequences

-Every definable set is Borel (but need to be locally closed)
-Each $\mathbb{A}_K$ is decidable (Weisspfenning, 1970's)

Example of a definable set not a finite union of locally closed sets

$$X = \{f : FIN([x^2 \neq x](f))\}$$

Let $X^{(1)} = fr(fr(X))$ (fr=frontier)

Then $X^{(1)} = X$, and result follows from work of Miller and Dougherty.

In fact, X is not in $F_\sigma \cap G_\delta$, by work of Hausdorff.

X is actually F_σ and not G_δ. The following *locates* definable sets in the bottom reaches of the Borel hierarchy

Basic definable sets and their places in Borel hierarchy

1. $\{\bar{f} : [[\Phi(\bar{\nu})]](\bar{f}) = 0\}$, Φ a ring formula, is a finite union of locally closed sets
2. Same with $[[\Phi(\bar{\nu})]](\bar{f}) = 1$.
3. $\{\bar{f} : FIN[[\Phi(\bar{\nu})]](\bar{f}) = 0\}$ is a countable union of locally closed sets
4. $\{\bar{f} : \neg FIN[[\Phi(\bar{\nu})]](\bar{f}) = 0\}$ is a countable intersection of locally closed sets

Computing measures - Case $K = \mathbb{Q}$

Fix $n > 0$.
Let X consist of the adeles f such that

$$|f(\mathbb{R})|_{\mathbb{R}} \leq 1 \quad \text{and} \quad 0 \leq \nu_p(f(p)) \leq n$$

at the primes.
Then the measure of X is $\frac{1}{\zeta(n+1)}$

For general **rectangles** as above, one must use the Denef-Loeser work on motivic integration (work in *slow* progress)

K and the ideles $\mathbb{I}_K$

The idele group is the group of units of $\mathbb{A}_K$. It is important to note that we do not give it the subspace topology, but rather the restricted product topology inside the product of the local units. In this way it becomes a locally compact topological group. K^* is naturally(diagonally) embedded as a subgroup of $\mathbb{I}_K$.In fact it is discretely embedded

Content

The content of an idele is the product of its local absolute values, with the normalization given earlier. The content is a continuous epimomorphism c to the multiplicative group of positive reals, and the kernel $\mathbb{I}_K^1$ is closed in both the ideles and the adeles. In the additive setting, $\mathbb{A}_K/K$ is compact. In the multiplicative setting K^* is in the kernel of c, and the quotient

$$\mathbb{I}_K^1/K^*$$

is compact.

Tamagawa numbers

We keep the discussion a bit vague, though the issue is a big one.Let G be a (suitably well-behaved) algebraic group defined over K. G need not be affine. One gives a natural meaning to $G(\mathbb{A}_K)$, and by algebraic -geometric and analytic means (going back to Tamagawa, Weil and others) defines a Haar measure on $G(\mathbb{A}_K)$ via local measures.The product involved is not always well-defined,e.g for the multiplicative group, but when it is the resulting measure is, in many fundamental cases, 1, and this generalizes very important work of Siegel on quadratic spaces. The point to be made here is that the Tamagawa number is simply the measure of a set definable or interpretable in the adeles, with respect to a natural measure. The scope of this phenomenon remains to be investigated. Incidentally, the Birch-SwinnertonDyer Conjecture emerges from similar considerations.

Enriched language

There is of course no possibility of interpreting , in a first-order way, K in $\mathbb{A}_K$, but it turns out that one can reach some interesting model theory by reflecting on definable supersets of K in $\mathbb{A}_K$. For example, any k in K^* ,for $K = \mathbb{Q}$ is a nonsquare at an even number of v (for this the presence of the v corresponding to R is crucial) because of quadratic reciprocity.Now, the property of an adele of being a nonsquare at finitely many v is first-order,by our work.This led us to reflect on a hybrid of our standard Boolean algebra set -up with Presburger arithmetic.

Greater Expressive Power

We noted (as others may well have over the last 60 years) that if we enrich the Boolean structure further by adding for $n \geq 2$ and $r \geq 0$ a predicate $FIN_{n,r}$ to mean *has cardinality congruent to r mod n* we still have quantifier elimination and decidability.

This gives the obvious corresponding results in the adelic situation. Though we have not carefully verified it in this situation, we expect that the extended formalism has more expressive power than the original ring formalism. The extended definable sets will still be Borel.

Primes and Minimal Idempotents

For simplicity, let K be $\mathbb{Q}$. We naturally associated the atoms of the Boolean algebra with minimal idempotents of the adeles, and with primes of K. In fact, using some basic algebra of finite fields one can give, for each prime (absolute value) p (including the real absolute value!) a formula of rings defining exactly the idempotent corresponding to p.Thus we have an elementary correspondence between minimal idempotents and primes p. Now, we simply ask.What are the definable sets of minimal idempotents in the adeles? The beautiful answer,using our work and that of Ax from the Annals 1968 is that the definable sets of nonarchimedean primes are exactly the sets corresponding to the sets in Ax's Boolean algebra of sets