Effective descriptive set theory

Yiannis N. Moschovakis
UCLA and University of Athens

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Outline

(I) A bit of history (3 slides)

(II) The basic notions (7 slides)

(III) Some characteristic effective results (6 slides)

(IV) HYP isomorphism and reducibility (Gregoriades) (3 slides)

- *Descriptive set theory*, ynm, 1980, Second Edition 2009
- *Classical descriptive set theory as a refinement of effective descriptive set theory*, ynm, 2010
- *Kleene's amazing second recursion theorem*, ynm, 2010
- *Notes on effective descriptive set theory*, ynm and Vassilios Gregoriades, in preparation

(The first three are posted on `www.math.ucla.edu/∼ynm`)

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The arithmetical hierarchy

- **Kleene [1943]**: The arithmetical hierarchy on subsets of $\mathbb{N}$

recursive $\subsetneq \Sigma^0_1$ (rec. enumerable) $\subsetneq \Sigma^0_2 \subsetneq \cdots$ $(\neg\Sigma = \Pi,\ \Sigma \cap \Pi = \Delta)$

- Tool for giving easy (semantic) proofs of Gödel's First Incompleteness Theorem, Tarski's Theorem on the arithmetical undefinability of arithmetical truth, etc.

- **Mostowski [1947]**: Reinvents the arithmetical hierarchy, using as a model the classical projective hierarchy on sets of real numbers

Borel $\subsetneq \boldsymbol{\Sigma}^1_1$ (analytic) $\subsetneq \boldsymbol{\Sigma}^1_2 \subsetneq \cdots$ $(\neg\boldsymbol{\Sigma} = \boldsymbol{\Pi},\ \boldsymbol{\Sigma} \cap \boldsymbol{\Pi} = \boldsymbol{\Delta})$

- He grounds the analogy on the two basic results

$$\text{Kleene: } \Delta^0_1 = \text{recursive}, \quad \text{Suslin: } \boldsymbol{\Delta}^1_1 = \text{Borel}$$

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- **Mostowski [1951]** introduces the hyperarithmetical hierarchy
  - In modern notation, roughly, he defines for each constructive ordinal $\xi < \omega_1^{\mathrm{CK}}$ a universal set for a class P_ξ of subsets of $\mathbb{N}$
  - The analogy now is between HYP and the Borel sets of reals. M. mimics closely Lebesgue's classical definition of $\boldsymbol{\Sigma}^0_\xi$ sets, replacing countable unions by projection along $\mathbb{N}$ and using effective diagonalization at limit ordinals
  - There are technical difficulties with the effective version. M. does not give detailed proofs and refers to the need for

    "a rather developed technique which we do not wish to presuppose here"

To make the definition precise, one needs effective transfinite recursion on ordinal notations.

This depends on the 2nd Recursion Theorem and was introduced in the literature by Kleene [1938], not cited in this paper.

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$$|a| = |b| + 1 \implies H_a = H_b' \text{ (the jump of } H_b)$$

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$$P_\xi = \text{ the class of sets recursively reducible to } H_a \ (|a| = \xi + 1)$$

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Recursive Polish metric spaces

- A metric space $(\mathcal{X}, d)$ is Polish if it is separable and complete
- A presentation of $\mathcal{X}$ is any pair $(d, \mathbf{r})$ where $\mathbf{r} : \mathbb{N} \to \mathcal{X}$ and $\mathbf{r}[\mathbb{N}] = \{r_0, r_1, \ldots\}$ is dense in $\mathcal{X}$
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$$P^{d,\mathbf{r}}(i,j,k) \iff d(r_i, r_j) \le q_k, \quad Q^{d,\mathbf{r}}(i,j,k) \iff d(r_i, r_j) < q_k,$$

are recursive, where $q_k = \frac{(k)_0}{(k)_1 + 1}$
- Recursive Polish metric space: $(\mathcal{X}, d, \mathbf{r})$ with recursive $(d, \mathbf{r})$
- $\mathbb{N} = \{0, 1, \ldots\}$ as a discrete space;
the reals $\mathbb{R}$ and Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ with their standard metrics;
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Open and effectively open (Σ^0_1) pointsets

Fix a recursive Polish metric space $(\mathcal{X}, d, \{r_0, r_1, \ldots\})$

- Codes of nbhds: For each $s \in \mathbb{N}$, let

$$N_s = N(\mathcal{X}, s) = \left\{ x \in \mathcal{X} : d(x, r_{(s)_0}) < q_{(s)_1} \right\}$$

- A pointset $G \subseteq \mathcal{X}$ is open if $G = \bigcup_n N_{\varepsilon(n)} = \bigcup_n N(\mathcal{X}, \varepsilon(n))$ with some $\varepsilon \in \mathcal{N}$ Any such ε is a code of G
- G is semirecursive or Σ^0_1 if it has a recursive code

Lemma (Normal Form for Σ^0_1)
$P \subseteq \mathcal{X}$, $Q \subseteq \mathcal{X} \times \mathcal{Y}$ are Σ^0_1 if and only if

$$\begin{aligned} P(x) &\iff (\exists s)[x \in N(\mathcal{X}, s) \;\&\; P^*(s)] \\ Q(x, y) &\iff (\exists s)(\exists t)[x \in N(\mathcal{X}, s) \;\&\; y \in N(\mathcal{Y}, t) \;\&\; Q^*(s, t)] \end{aligned}$$

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Recursive Polish spaces

Def A topological space $(\mathcal{X}, \mathcal{T})$ is Polish if there is a d such that $(\mathcal{X}, d)$ is a Polish metric space which induces $\mathcal{T}$

Def A recursive Polish space is a set $\mathcal{X}$ together with a family $\mathcal{R} = \mathcal{R}(\mathcal{X})$ of subsets of $\mathbb{N} \times \mathcal{X}$ such that for some $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and $\mathbf{r} : \mathbb{N} \to \mathcal{X}$ the following conditions hold:

(RP1) $(\mathcal{X}, d, \mathbf{r})$ is a recursive Polish metric space, and

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The analogies between the classical and the effective theory

- A pointset is any subset $P \subseteq \mathcal{X}$ of a recursive Polish space, formally a pair $(P, \mathcal{X})$
- A pointclass is any collection Γ of pointsets, e.g.,

$$\Sigma^0_1 = \text{the semirecursive pointsets}, \quad \Pi^0_1 = \{\mathcal{X} \setminus P : P \in \Sigma^0_1\},$$

$$\boldsymbol{\Sigma}^0_1 = \text{the open pointsets} \supseteq \Sigma^0_1, \quad \boldsymbol{\Pi}^0_1 = \text{all closed pointsets} \supseteq \Pi^0_1$$

- The arithmetical and analytical pointclasses

$$\Pi^0_k = \neg\Sigma^0_k, \ \Sigma^0_{k+1} = \exists^{\mathbb{N}}\Pi^0_k, \quad \Delta^0_k = \Sigma^0_k \cap \Pi^0_k$$

$$\Sigma^1_1 = \exists^{\mathcal{N}}\Pi^0_1, \quad \Pi^1_k = \neg\Sigma^0_k, \ \Sigma^1_{k+1} = \exists^{\mathcal{N}}\Pi^1_k, \ \Delta^1_k = \Sigma^1_k \cap \Pi^1_k$$

- The finite Borel and projective pointclasses

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Hyperarithmetical as effective Borel

For $A \subseteq \mathcal{X}$:

- **Def** α is a K_ξ-code of A : $\alpha \in \mathsf{K}_\xi$ & $A = B^{\mathcal{X}}_\alpha$
  - $A \in \mathbf{\Sigma}^0_\xi \iff A$ has a K_ξ-code
- **Def** α is a Borel code of A : $\alpha \in \mathsf{K}$ & $A = B^{\mathcal{X}}_\alpha$
  - A is Borel $\iff A$ has a Borel code
- **Def** A is HYP $\iff A$ has a recursive Borel code
- **Def** $f : \mathcal{X} \to \mathcal{Y}$ is HYP if $\{(x, s) : f(x) \in N(\mathcal{Y}, s)\} \in$ HYP
- **Def** (Louveau 1980) $A \in \Sigma^0_\xi \iff A$ has a recursive K_ξ-code

- For $\mathcal{X} = \mathcal{Y} = \mathbb{N}$, these definitions of HYP agree with the classical ones
- The pointclasses Σ^0_ξ stabilize for $\xi \geq \omega_1^{\mathrm{CK}}$, and for $\xi < \omega_1^{\mathrm{CK}}$ on $\mathbb{N}$ they are (essentially) those defined by Mostowski and Kleene
- $A \subseteq \mathcal{X}$ is Borel exactly when it is HYP(α) for some $\alpha \in \mathcal{N}$
- $f : \mathcal{X} \to \mathcal{Y}$ is Borel (measurable) exactly when it is HYP(α) for some $\alpha \in \mathcal{N}$

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Partial functions and potential recursion

- A partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is potentially recursive if there is a Σ^0_1 pointset $P \subseteq \mathcal{X} \times \mathbb{N}$ which computes f on its domain, i.e.,

$$f(x)\downarrow \implies \Big(f(x) \in N(\mathcal{Y}, s) \iff P(x, s)\Big) \qquad (*)$$

Canonical Extension Theorem (Eff. version of classical fact)

Every potentially recursive $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ has a potentially recursive extension $\overline{f} \supseteq f$ whose domain is Π^0_2

Refined Embedding Theorem (Eff. version of classical fact)

For every recursive Polish $\mathcal{X}$, there is a (total) recursive surjection

$$\pi : \mathcal{N} \twoheadrightarrow \mathcal{X}$$

and a Π^0_1 set $A \subseteq \mathcal{N}$ such that π is injective on A and $\pi[A] = \mathcal{X}$

- *If $f : \mathcal{N} \rightharpoonup \mathcal{N}$ and $f(\alpha)\downarrow$, then $f(\alpha)$ is recursive in α*
- Every potentially recursive function is continuous on its domain; and every $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ which is continuous on its domain is potentially ε-recursive for some $\varepsilon \in \mathcal{N}$

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The Suslin-Kleene Theorem

Theorem

(a) *Every Borel pointset is* $\mathbf{\Delta}^1_1$, *uniformly*

(b) *Every* $\mathbf{\Delta}^1_1$ *pointset is Borel, uniformly*

The precise version of (b): *For some potentially recursive* $\mathbf{u} : \mathcal{N} \rightharpoonup \mathcal{N}$, if α is a $\mathbf{\Delta}^1_1$-code of some $A \subseteq \mathcal{X}$,

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- Suslin's Theorem: $\mathbf{\Delta}^1_1 = $ Borel
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- Both proofs use effective transfinite recursion and (a) is routine (b) uses the fact that a classical proof of Suslin's Theorem (in Kuratowski) is constructive (cf. Kleene's realizability theory)
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- Kleene's Theorem: On $\mathbb{N}$, $\Delta^1_1 =$ HYP
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The Suslin-Kleene Theorem

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(a) *Every Borel pointset is* $\mathbf{\Delta}^1_1$, *uniformly*
(b) *Every* $\mathbf{\Delta}^1_1$ *pointset is Borel, uniformly*

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The effective Perfect Set Theorem

Theorem (Harrison)
If $A \subseteq \mathcal{X}$ is Σ^1_1 and has a a member $x \in A$ which is not HYP, *then A has a perfect subset*

Corollary (Suslin)
Every uncountable $\boldsymbol{\Sigma}^1_1$ pointset has a perfect subset

- Suslin's Perfect Set Theorem followed earlier results of Hausdorff and Alexandrov for Borel sets and was very important for the classical theory: it implies that the Continuum Hypothesis holds for $\boldsymbol{\Sigma}^1_1$ (analytic) sets
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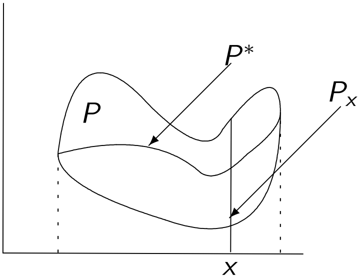
The HYP Uniformization Criterion



P^* uniformizes P.

Theorem

A pointset $P \subseteq \mathcal{X} \times \mathcal{Y}$ in HYP *can be uniformized by a* HYP *set P^* if and only if for every $x \in \mathcal{X}$,*

$$(\exists y)P(x,y) \iff (\exists y \in \mathrm{HYP}(x))P(x,y)$$

- (Classical) *If every section of a Borel set $P \subseteq \mathcal{X} \times \mathcal{Y}$ is countable, then P has a Borel uniformization*

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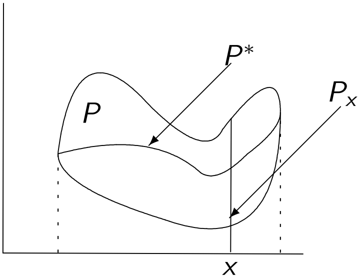

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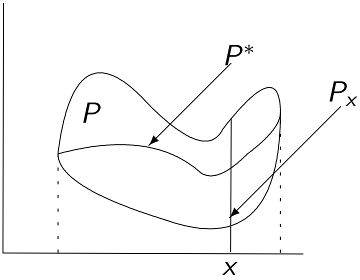

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Louveau's Theorem

Theorem (Louveau 1980)

For every $\mathcal{X}$, every $P \subseteq \mathcal{X}$ and every recursive ordinal ξ

$$P \in (\mathsf{HYP} \cap \boldsymbol{\Sigma}^0_\xi) \iff P \in \Sigma^0_\xi(\alpha) \text{ for some } \alpha \in \mathsf{HYP}$$

- This is a basic result about the (relativized) effective hierarchies $\Sigma^0_\xi(\alpha)$ and has many classical and effective applications (including some more detailed versions of the results in the last three slides)

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The Harrington-Kechris-Louveau Theorem

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For every HYP *equivalence relation E on a recursive Polish space $\mathcal{X}$:*

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Luzin's favorite characterization of the Borel sets

Theorem
A set $A \subseteq \mathcal{X}$ is HYP *if and only if A is the recursive, injective image of a Π^0_1 subset of $\mathcal{N}$*

Corollary (Luzin)
A set $A \subseteq \mathcal{X}$ is Borel if and only if A is the continuous, injective image of a closed subset of $\mathcal{N}$

• Luzin's proof is not difficult, so the effective version does not contribute much beyond the stronger statement However, by its proof:

Theorem (ynm 1973)
Assume $\mathbf{\Sigma}^1_2$-determinacy A set $A \subseteq \mathcal{X}$ is Δ^1_3 if and only if A is the recursive, injective image of a Π^1_2 subset of $\mathcal{N}$

• The effective theory is indispensable in the study of projective sets under strong, set theoretic hypotheses

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HYP-recursive (eff. Borel) functions and isomorphisms

- *Every uncountable Polish space $\mathcal{X}$ is Borel isomorphic with $\mathcal{N}$*
- Thm [G] *There exist uncountable recursive Polish spaces which are not* HYP*-isomorphic with $\mathcal{N}$*
- A (total) function $f : \mathcal{X} \to \mathcal{Y}$ is HYP(ε)-recursive if it is computed by a HYP(ε) relation, i.e.,

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- The local space parameter For any space $\mathcal{X}$, put

$$P_{\mathcal{X}} = \{s \in \mathbb{N} : N(\mathcal{X}, s) \text{ is uncountable}\}$$

  - $P_{\mathcal{X}}$ is Σ^1_1;
    it is recursive if $\mathcal{X}$ perfect;
    and for some $\mathcal{X}$ it is Σ^1_1-complete

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HYP-recursive (eff. Borel) functions and isomorphisms

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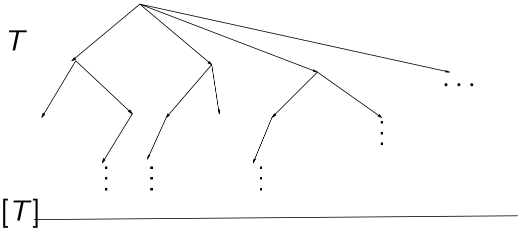

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$$\mathcal{N}^T = T \cup [T]$$

with the natural metric, so that $\lim_n(\alpha(0), \ldots, \alpha(n)) = \alpha$

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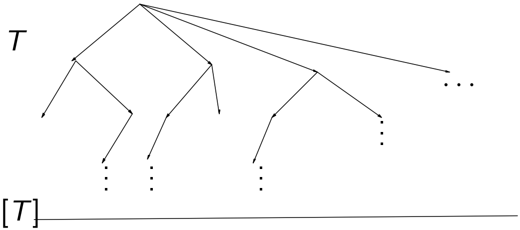

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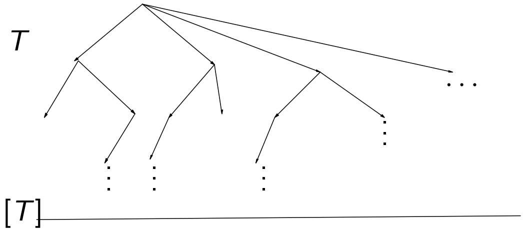

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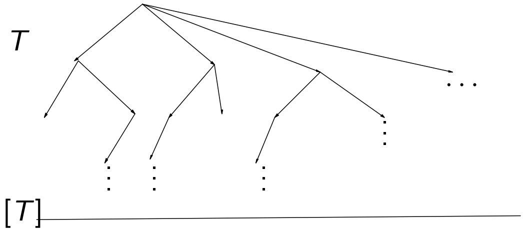

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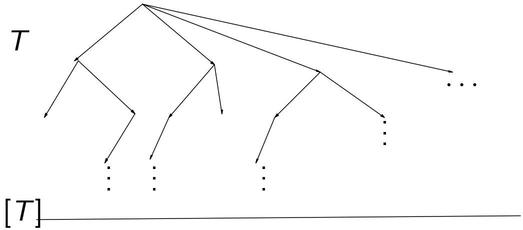

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Recursive Polish spaces under $\preceq_{\mathrm{HYP}}$

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- $\mathcal{N}^T$ is a Kleene space if $[T]$ is not empty and has no HYP branches

Theorem (G)

(a) *Every Kleene space occurs in an infinite* $\preceq_{\mathrm{HYP}}$*-antichain of Kleene spaces*

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