General approach to Ramsey theory and a new Ramsey theorem

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Outline of Topics

1. Introduction
2. Dual Ramsey theorem for trees
3. Algebraic notions
4. Abstract Ramsey and abstract pigeonhole statements

Introduction

For $n \in \mathbb{N}$, put

$$[n] = \{1, 2, \ldots, n\};$$

in particular,

$$[0] = \emptyset.$$

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Later, close connections between extreme amenability and finite Ramsey theory were discovered.

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— A is linearly ordered;

— the class of all finite substructures of A is Ramsey.

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monochromatic.

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Prömel–Voigt: Rigid surjections from $[n]$ to $[m]$ are in a bijective correspondence with m-partitions of n:

$$s \to \mathcal{P}_s = \{s^{-1}(i)\colon i \in [m]\}.$$

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J. Moore: a Ramsey statement equivalent to amenability of Thompson's group

D. Bartošova–A. Kwiatkowska: a Ramsey statement needed for dynamics (computation of the universal minimal flow) of the homeomorphism group of the Lelek fan

Dual Ramsey theorem for trees

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s is **dual to** i if for each $w \in T$

$$s([w]) = i^{-1}([w]).$$

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In the case when S and T are linear orders $[k]$ and $[l]$, the new notion of rigid surjection coincides with the old one.

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Algebraic notions

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Most finite unstructured Ramsey theorems are special instances of the above theorem. The theorem also makes it possible to prove new results.

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Normed composition spaces

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- $|\cdot|$ is a function from A to a partially ordered set $(L, \leq)$ (**norm**).

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(iii) if $|b| \leq |c|$ and $a \cdot c$ is defined, then so is $a \cdot b$ and $|a \cdot b| \leq |a \cdot c|$.

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Let $w \in T$, $f \colon T^w \to S$ and $g \colon V \to T$. Define

$$g \cdot f = f \circ g^w.$$

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otherwise, let v be the second $\leq_S$-largest vertex in S, and let

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A **with** $\cdot$**,** ∂**,** $|\cdot|$ **defined above is a normed composition space.**

Lifting multiplication to sets

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$$F \bullet G = F \cdot G.$$

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$\mathcal{F}$ is a family of sets over A.

Abstract Ramsey and abstract pigeonhole statements

Ramsey statement

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Dual trees (ctd)

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Condition (R) is the dual Ramsey theorem for trees with **sealed** rigid surjections.

Pigeonhole statement

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Price: make f behave as prescribed by some $a \in A$ on a part of A

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Dual trees (ctd)

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So dual Ramsey theorem for trees **holds**.