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On ideal equal convergence

Joint work with Rafał Filipów

1. Introduction

Let f_n $(n \in \mathbb{N})$ and f be real-valued functions defined on a set X. We say that a sequence (f_n) is *equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \to 0$ such that for every $x \in X$ there is N with $|f_n(x) - f(x)| < \varepsilon_n$ for every $n > N$.

The notion of equal convergence was introduced by Császár and Laczkovich. It is known that equal convergence is weaker than uniform convergence and stronger than pointwise convergence i.e. if (f_n) is uniformly convergent to f then (f_n) is equally convergent to f; and if (f_n) is equally convergent to f then (f_n) is pointwise convergent to f.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. We say that a sequence of reals (x_n) is $\mathcal{I}$*-convergent* to $x \in \mathbb{R}$ if $\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$ (Kostyrko, Šalát, Wilczyński). We write $(x_n) \xrightarrow{\mathcal{I}} x$ in this case.

The notion of equal convergence was generalized with the aid of ideals on $\mathbb{N}$ in two different ways. Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$.

- P. Das, S. Dutta, S. K. Pal says that (f_n) is $\mathcal{I}$*-equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{I}} 0$ such that $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$
- R. Filipów, P. Szuca says that (f_n) is $\mathcal{I}$*-equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \to 0$ such that $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$.

The only difference between these two definitions of ideal equal convergence is the requirement that the sequence (ε_n) is either convergent (in the classical meaning) to zero or it is convergent to zero with respect to ideal $\mathcal{I}$. It is easy to see that if (f_n) is $\mathcal{I}$-equally convergent to f in the sense of Filipów, Szuca, then it is also

$\mathcal{I}$-equally convergent to f in a sense of Das, Dutta, Pal, when we assume that $\mathcal{I}$ contains all finite sets.

An ideal is called *countably generated* if there are sets $A_1, A_2, \dots \in \mathcal{I}$ such that for every $A \in \mathcal{I}$ there is $n \in \mathbb{N}$ with $A \subseteq A_n$.

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. An ideal $\mathcal{I}$ is a $P(\mathcal{J})$*-ideal* if for any sets $A_1, A_2, \dots \in \mathcal{I}$ there is a set $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathcal{J}$ for every $n \in \mathbb{N}$ (M. Maćaj, M. Sleziak). Note that $P(\mathrm{Fin})$-ideals are also called P*-ideals.* It is easy to see that every ideal $\mathcal{I}$ is always $P(\mathcal{I})$-ideal.

2. Ideal equal convergence

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let f_n $(n \in \mathbb{N})$ and f be real-valued functions defined on a set X. We say that the sequence (f_n) is $(\mathcal{I}, \mathcal{J})$*-equally convergent* to f if there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$. We write $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ in this case.

It is easy to see that ideal equal convergence introduced by Das, Dutta, Pal is equivalent to our $(\mathcal{I}, \mathcal{I})$-equal convergence, and ideal equal convergence introduced by Filipów, Szuca is equivalent to our $(\mathcal{I}, \mathrm{Fin})$-equal convergence.

Proposition 2.1. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.*
2. *$\mathcal{J} \subseteq \mathcal{I}$.*

Corollary 2.2. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let (f_n) be a sequence of real-valued functions defined on a set X. If $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.*

Proposition 2.3. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

1. *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*
2. *$\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

Corollary 2.4. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ then $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

3. Ideal convergence: uniform, equal and pointwise

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence $(f_n)_{n\in\omega}$ is

- *$\mathcal{I}$-uniformly convergent* to f if for every $\varepsilon > 0$ the set $\{n : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } x\} \in \mathcal{I}$. We write $(f_n)_n \xrightarrow{\mathcal{I}-u} f$ for short.
- *$\mathcal{I}$-pointwise convergent* to f if for every $\varepsilon > 0$ and every $x \in X$ the set $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$. We write $(f_n)_n \xrightarrow{\mathcal{I}} f$ for short.

Das, Dutta, Pal proved that if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.

Proposition 3.1. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*
(2) *$\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

Corollary 3.2. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

Proposition 3.3. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*
(2) *$\mathcal{J} \subseteq \mathcal{I}$.*

Corollary 3.4. *Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. Let (f_n) be a sequence of real-valued functions defined on a set X.*

(1) *If* $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ *then* $(f_n) \xrightarrow{\mathcal{I}} f$.
(2) *If* $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ *then* $(f_n) \xrightarrow{\mathcal{I}} f$.

Corollary 3.5. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}-u} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ and if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*
(2) *$\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

4. Ideal σ-uniform convergence

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. A sequence (f_n) of real-valued functions defined on X is $\sigma - \mathcal{I}$*-uniformly convergent* to $f : X \to \mathbb{R}$ if there are sets $X_k \subseteq X$ $(k \in \mathbb{N})$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $(f_n \restriction X_k) \xrightarrow{\mathcal{I}-u} f \restriction X_k$ for every $k \in \mathbb{N}$. We write $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$ in this case.

It is easy to see that for every ideal $\mathcal{I}$, if $(f_n) \xrightarrow{\mathcal{I}-u} f$, then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$; and if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, then $(f_n) \xrightarrow{\mathcal{I}} f$.

Császár and Laczkovich proved that the equal convergence and σ-uniform convergence are the same i.e. $(f_n) \xrightarrow{e} f \iff (f_n) \xrightarrow{\sigma-u} f$. Below we examine the relationship between ideal equal convergence and ideal σ-uniform convergence.

Das, Dutta, Pal proved that for every ideal $\mathcal{I}$ if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$.

Proposition 4.1. *Let X be a nonempty set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*
(2) *$\mathcal{I}$ is a $P(\mathcal{J})$-ideal.*

Corollary 4.2. *Let X be a nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$, then $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

Proposition 4.3. *Let $|X| \geq \mathfrak{c}$. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.*
(2) *$\mathcal{I}$ is countably generated and $\mathcal{J} \subseteq \mathcal{I}$.*

Corollary 4.4. *Let $|X| \geq \mathfrak{c}$. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{I})-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.*
(2) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},Fin)-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.*
(3) *$\mathcal{I}$ is countably generated.*

Proposition 4.5. *Let X be a countable set. Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}-u} f$.*
(2) *$\mathcal{J} \subseteq \mathcal{I}$.*

5. Filter convergence: uniform, equal, σ-uniform and pointwise

Let $\mathcal{I}, \mathcal{J}$ be ideals on $\mathbb{N}$. A sequence (f_n) of real-valued functions defined on X is

- $\mathcal{I}^*$-*uniformly convergent* to $f : X \to \mathbb{R}$ if there exists a set $F \in \mathcal{I}^*$ with $(f_n)_{n \in F} \xrightarrow{u} f$. We write $(f_n) \xrightarrow{\mathcal{I}^*-u} f$ in this case.
- $\mathcal{I}^*$-*pointwise convergent* to $f : X \to \mathbb{R}$ if there exists a set $F \in \mathcal{I}^*$ with $(f_n)_{n \in F} \to f$. We write $(f_n) \xrightarrow{\mathcal{I}^*} f$ in this case.
- $(\mathcal{I}^*, \mathcal{J})$-*equally convergent* to $f : X \to \mathbb{R}$ if there exists a set $F \in \mathcal{I}^*$ with $(f_n)_{n \in F} \xrightarrow{(\mathrm{Fin},\mathcal{J})-e} f$. We write $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ in this case.
- $\sigma - \mathcal{I}^*$-*uniformly convergent* to $f : X \to \mathbb{R}$ if there exist X_k $(k \in \mathbb{N})$ with $X = \bigcup_{k \in \mathbb{N}} X_k$ and $(f_n \restriction X_k) \xrightarrow{\mathcal{I}^*-u} f \restriction X_k$ for every $k \in \mathbb{N}$. We write $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$ in this case.

The notions of $\mathcal{I}^*$-uniform, $\mathcal{I}^*$-pointwise and $\sigma - \mathcal{I}^*$-uniform convergence were introduced by Das, Dutta, Pal. They introduced also $\mathcal{I}^*$-equal convergence, which is equivalent to $(\mathcal{I}^*, \mathrm{Fin})$-equal convergence in our notation.

Proposition 5.1. *Let X be nonempty set. Let $\mathcal{I},\mathcal{J}$ be ideals on $\mathbb{N}$. Then for every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\mathcal{I}^*-u} f$ then $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$.*

Proposition 5.2. *Let X be nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{J}$ be a P-ideal. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}^*} f$.*
(2) *$\mathcal{J} \subseteq \mathcal{I}$.*

Proposition 5.3. *Let X be nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $\mathcal{J}$ be a P-ideal. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$.*
(2) *$\mathcal{J} \subseteq \mathcal{I}$.*

Proposition 5.4. *Let X be nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$, $\mathcal{J}$ be a P-ideal and $\mathcal{J} \subseteq \mathcal{I}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$ then $(f_n) \xrightarrow{(\mathcal{I}^*,\mathcal{J})-e} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*

Proposition 5.5. *Let X be nonempty set. Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence (f_n) of real-valued functions defined on a set X, if $(f_n) \xrightarrow{\sigma-\mathcal{I}^*-u} f$ then $(f_n) \xrightarrow{\mathcal{I}^*} f$.*
(2) *$\mathcal{I}$ is a P-ideal.*