The Mostowski Collapse and the Inner Model Program

W. Hugh Woodin

University of California, Berkeley

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The Mostowski Collapse

Theorem

Suppose M is a transitive set and $X \prec M$. Then there is a unique transitve set N and isomorphism

$$\pi : N \cong X.$$

The generalizations of the Mostowski Collapse are ubiquitous in Set Theory.

The Universe of Sets

The power set

Suppose X is a set. The powerset of X is the set

$$\mathcal{P}(X) = \{Y \mid Y \text{ is a subset of } X\}.$$

Cumulative Hierarchy of Sets

The universe V of sets is generated by defining V_α by induction on the ordinal α:

1. $V_0 = \emptyset$,
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
3. *if α is a limit ordinal then* $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

- Every set belongs to V_α for some ordinal α.

Logical definability from parameters

Definition

Suppose X is a transitive set. A subset $Y \subseteq X$ is logically definable in $(X, \in)$ from parameters if for some formula $\varphi[x_0, \ldots, x_n]$ and for some parameters $a_1, \ldots, a_n \in X$,

$$Y = \{a \in X \mid (X, \in) \models \varphi[a, a_1, \ldots, a_n]\}$$

The definable power set

For each set X, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that X is logically definable in the structure $(X, \in)$ from parameters in X.

- (Axiom of Choice) $\mathcal{P}_{\mathrm{Def}}(X) = \mathcal{P}(X)$ if and only if X is finite.
- $\mathcal{P}_{\mathrm{Def}}(V_{\omega+1}) \cap \mathcal{P}(\mathbb{R})$ is exactly the projective sets.

The effective cumulative hierarchy: L

Gödel's constructible universe, L

Define L_α by induction on α as follows.

1. $L_0 = \emptyset$,
2. (Successor case) $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$,
3. (Limit case) $L_\alpha = \cup\{L_\beta \mid \beta < \alpha\}$.

L is the class of all sets X such that $X \in L_\alpha$ for some ordinal α.

Theorem (Gödel)

Suppose $X \prec L_\alpha$. Then there is a unique ordinal β and isomorphism

$$\pi : L_\beta \cong X.$$

Theorem (Scott)

Assume $V = L$. Suppose M is a transitive set and that

$$X \prec M$$

is an elementary substructure such that $X \cong V_\alpha$ for some α. Then $V_\alpha = X$.

Axioms which assert the existence of $X \prec M$ where M is transitive,

$$X \cong V_\alpha$$

and $X \neq V_\alpha$ yield the modern hierarchy of large cardinal axioms.

- *These axioms imply $V \neq L$.*

Strong axioms of infinity: large cardinal axioms

Basic template for large cardinal axioms

A cardinal κ is a large cardinal *if there exists an elementary embedding,*

$$j : V \to M$$

such that M is a transitive class and κ is the least ordinal such that $j(\alpha) \neq \alpha$.

- Requiring M be *close* to V yields a hierarchy of large cardinal axioms:
    - simplest case is where κ is a *measurable cardinal*.
- $M = V$ contradicts the Axiom of Choice.

The Inner Model Program seeks enlargements of L in which these large cardinals can exist.

- *The problem becomes more difficult as one ascends the hierarchy.*

The hierarchy of large cardinal axioms–short version

- *There is a proper class of measurable cardinals*.
- *There is a proper class of strong cardinals*.
- *There is a proper class of Woodin cardinals*.
- *There is a proper class of superstrong cardinals*.

. . . .

- *There is a proper class of supercompact cardinals*.
- *There is a proper class of extendible cardinals*.
- *There is a proper class of huge cardinals*.
- *There is a proper class of ω-huge cardinals*.

Enlargements of L

Definition

Suppose $\mathbb{E}$ is a set (or class). Then

1. $L_0[\mathbb{E}] = \emptyset$,
2. (Successor case) $L_{\alpha+1}[\mathbb{E}] = \mathcal{P}_{\mathrm{Def}}(Z)$ where
$$Z = L_\alpha[\mathbb{E}] \cup \{\mathbb{E} \cap L_\alpha[\mathbb{E}]\},$$
3. (Limit case) $L_\alpha[\mathbb{E}] = \cup\{L_\beta[\mathbb{E}] \mid \beta < \alpha\}$.

- $L[\mathbb{E}]$ is the class of all sets X such that $X \in L_\alpha[\mathbb{E}]$ for some ordinal α.

- If $\mathbb{E} \cap L = \emptyset$ then $L[\mathbb{E}] = L$.
- For every set X there is a set $\mathbb{E}$ such that $X \in L[\mathbb{E}]$.
  - This is equivalent to the Axiom of Choice.

The building blocks for inner models: Extenders

Suppose that

$$j : V \to M$$

is an elementary embedding with critical point κ, $\kappa < \eta$, and that

$$\mathcal{P}(\eta) \subset M.$$

The (strong) extender E of length η derived from j

The extender *E of length η defined from j is the function:*

$$E : \mathcal{P}(\eta) \to \mathcal{P}(\eta)$$

where $E(A) = j(A) \cap \eta$.

Two ordinals associated to the extender E:

- $\mathrm{CRT}(E) = \min\{\alpha \mid E(\alpha) \neq \alpha\} = \kappa$.
- $\mathrm{LTH}(E) = \eta$ where $\mathrm{dom}(E) = \mathcal{P}(\eta)$.

Large cardinal axioms in terms of extenders

δ is a strong cardinal if

- *for each $\gamma > \delta$ there exists an extender E such that $\mathrm{CRT}(E) = \delta$ and $\mathrm{LTH}(E) \geq \gamma$.*

δ is a supercompact cardinal if

- *for each $\gamma > \delta$ there exists an extender E such that $E(\mathrm{CRT}(E)) = \delta$ and $\mathrm{LTH}(E) \geq \gamma$.*

δ is an extendible cardinal if

- *for each $\gamma > \delta$ there exists an extender E such that $\mathrm{CRT}(E) = \delta$, $E(\delta) > \gamma$, and $\mathrm{LTH}(E) > E(\gamma)$.*

Weak extender models and extender models

For a large cardinal axiom Φ:

Definition

A transitive class N is a **weak extender model for Φ** if Φ is witnessed to hold in N by extenders E of N such that

$$E = F|N$$

for some extender F.

- If Φ holds in V then V is a weak extender model for Φ.

Definition

A transitive class N is an **extender model for Φ** if for some sequence $\mathbb{E}$ of extenders:

1. $N = L[\mathbb{E}]$,
2. N is a weak extender model for Φ and this is witnessed by the extenders on the sequence $\mathbb{E}$.

The Inner Model Program

For a large cardinal axiom Φ and extender models, the simplest goal of the Inner Model Program is to answer the question:

Question

Assume that Φ holds. Must there exist an extender model N for Φ such that $N \neq V$?

Theorem (Martin-Steel)

Suppose there is a proper class of Woodin cardinals. Then there is an extender model N for a proper class of Woodin cardinals such that $N \neq V$.

Theorem (Martin-Steel)

Suppose there is a proper class of superstrong cardinals and the Iteration Hypothesis holds. Then there is an extender model N for a proper class of superstrong cardinals such that $N \neq V$.

Beyond superstrong: the Universality Theorem

Theorem (Universality Theorem)

Suppose that N is a weak extender model for δ is supercompact. Suppose that F is an extender such that:

- $\mathrm{CRT}(F) \geq \delta$ *and N is closed under F.*

Then $F|N \in N$.

- For any extender F, L is closed under F but $F|L \notin L$.
- **Any** weak extender model for δ is supercompact *inherits* all large cardinals from V which occur above δ.

Conclusion

The extension of the Inner Model Program to the level of one supercompact cardinal must yield the ultimate inner model

- *it must yield an ultimate version of L.*

Gödel's transitive class HOD

- For each set X, $\mathrm{TC(X)}$ is the smallest transitive set M with $X \in M$.

Definition

For each ordinal α, $\mathrm{HOD}_{\alpha+1}$ is the set of all sets $X \subseteq V_\alpha$ such that:

1. X is definable in V_α from ordinal parameters.
2. If $Y \in \mathrm{TC(X)}$ then Y is definable in V_α from ordinal parameters.

- The definition of $\mathrm{HOD}_{\alpha+1}$ is a mixture of the definition of $L_{\alpha+1}$ and $V_{\alpha+1}$.

Definition (Gödel)

HOD is the class of all sets X such that $X \in \mathrm{HOD}_{\alpha+1}$ for some α.

What about extender models for supercompact cardinals?

Definition

Suppose that $\mathbb{E} = \langle \mathbb{E}_\alpha : \alpha \in \mathrm{Ord} \rangle$ is a sequence.

Then $\mathbb{E}$ is **weakly Σ_2-definable** if there is a formula $\varphi(x)$ such that for all $\beta \in \mathrm{Ord}$,

- for all $\beta < \eta_1 < \eta_2 < \eta_3$, if

$$(\mathbb{E})^{V_{\eta_1}} | \beta = (\mathbb{E})^{V_{\eta_3}} | \beta$$

then $(\mathbb{E})^{V_{\eta_1}} | \beta = (\mathbb{E})^{V_{\eta_2}} | \beta = (\mathbb{E})^{V_{\eta_3}} | \beta$.

where $(\mathbb{E})^{V_\gamma} = \{a \in V_\alpha \mid V_\gamma \models \varphi[a]\}$.

- The sequence $\langle \mathrm{HOD} \cap V_\alpha : \alpha \in \mathrm{Ord} \rangle$ is weakly Σ_2-definable.

A serious obstruction

- Assume there is a proper class of supercompact cardinals

By class forcing one can arrange that the following hold

1. *$V = \mathrm{HOD}$ and there is a proper class of supercompact cardinals.*
2. *Suppose $\mathbb{E}$ is an extender sequence such that*
   (a) *$L[\mathbb{E}]$ is an extender model for δ is a supercompact cardinal,*
   (b) *$\mathbb{E}$ is weakly Σ_2-definable.*

   Then $V \subseteq L[\mathbb{E}]$.

Ramifications

Rules out developing the Inner Model Program to the level of constructing extender models for δ is supercompact.

- In fact one cannot go beyond the Martin-Steel extender models in any essential way.

Partial-extenders and partial-extender models

A **partial-extender** E *of length* η *is obtained from an elementary embedding*

$$j : N \to M$$

where $N \cap \mathcal{P}(\eta) = M \cap \mathcal{P}(\eta)$*:*

1. E *has domain* $N \cap \mathcal{P}(\eta)$*;*
2. $E(A) = j(A) \cap \eta$*.*

Definition

A transitive class N is a **partial-extender model for** Φ if for some sequence $\mathbb{E}$ of partial-extenders:

1. $N = L[\mathbb{E}]$,
2. N is a weak extender model for Φ and this is witnessed by the partial extenders on the sequence $\mathbb{E}$.

Good partial-extender models

- Every weak extender model can be re-organized as a partial-extender model, therefore:
  - Require a generalization of the Mostowski Collapse.

Definition

Suppose $L[\mathbb{E}]$ is a partial-extender model. Then $L[\mathbb{E}]$ is a **good partial-extender model** if for all $\eta < \alpha$, if

$$X \prec (L_\alpha[\mathbb{E}], \mathbb{E} \cap L_\alpha[\mathbb{E}])$$

is the elementary substructure given by the elements which are definable with parameters from η then

$$X \cong (L_\beta[\mathbb{E}], \mathbb{E} \cap L_\beta[\mathbb{E}])$$

for some β.

- If $L[\mathbb{E}]$ is a good partial-extender model then the Generalized Continuum Hypothesis holds in $L[\mathbb{E}]$.

Mitchell-Steel models

- The basic framework for good partial-extenders models for large cardinals up to the level of superstrong cardinals originates in the constructions of Mitchell and Steel.
  - There is an important variation due to Jensen which is equivalent but yields models with stronger condensation properties.

Theorem (Mitchell-Steel et al)

Assume there is a proper class of Woodin cardinals. Then there is a partial-extender model $L[\mathbb{E}]$ for a proper class of Woodin cardinals such that

(1) $\mathbb{E}$ *is weakly* Σ_2*-definable,*

(2) $L[\mathbb{E}]$ *is a good partial-extender model.*

Theorem (Mitchell-Steel et al)

Assume the Iteration Hypothesis and that there is a proper class of superstrong cardinals. Then there is a partial-extender model $L[\mathbb{E}]$ for a proper class of superstrong cardinals such that

(1) $\mathbb{E}$ *is weakly* Σ_2*-definable,*

(2) $L[\mathbb{E}]$ *is a good partial-extender model.*

Conjecture

Assume the Iteration Hypothesis and that there is an extendible cardinal. Then there is a partial-extender model $L[\mathbb{E}]$ for a supercompact cardinal such that

(1) $\mathbb{E}$ *is weakly* Σ_2*-definable,*

(2) $L[\mathbb{E}]$ *is a good partial-extender model.*

A first step

Theorem

Assume there is a supercompact cardinal and that the Iteration Hypothesis holds. Then there is a partial-extender model $L[\mathbb{E}]$ such that

(1) $\mathbb{E}$ *is weakly* Σ_2*-definable,*

(2) $L[\mathbb{E}]$ *is a good partial-extender model.*

(3) $L[\mathbb{E}]$ *is a weak extender model for the existence of* κ *such that* κ *is* κ^{+n}*-supercompact for all* $n < \omega$*.*

- *The theorem shows that the obstructions can be successfully dealt with.*
- *The constructions seem to indicate how to handle the general case.*

The Generic-Multiverse

Definition

Suppose that M is a countable transitive set and that

$$M \models \mathrm{ZFC}.$$

The **generic-multiverse** generated by M is the smallest set $\mathbb{V}_M$ of countable transitive sets such that for all pairs (N_0, N_1) of countable transitive sets if

1. N_1 is a generic extension of N_0
2. either $N_0 \in \mathbb{V}_M$ or $N_1 \in \mathbb{V}_M$

then both $N_0 \in \mathbb{V}_M$ and $N_1 \in \mathbb{V}_M$.

(meta) Definition

The **Generic-Multiverse** *is the generic-multiverse generated by V.*

Mitchell-Steel models and the Generic-Multiverse

Lemma ($V = L$)

V is the minimum universe of the Generic-Multiverse.

Theorem

Suppose $L[\mathbb{E}]$ is an (iterable) Mitchell-Steel model and

$$L[\mathbb{E}] \models \text{“There is a Woodin cardinal”}.$$

Then there is a Mitchell-Steel model $L[\mathbb{F}] \subset L[\mathbb{E}]$ such that $L[\mathbb{E}]$ is a generic extension of $L[\mathbb{F}]$.

The same theorem applies to the extension of Mitchell-Steel models beyond superstrong.

Is $\mathrm{Ultimate}\text{-}L$ a generalized Mitchell-Steel model?

Assume the Iteration Hypothesis holds in V and that there is a proper class of measurable Woodin cardinals.

- It is not known if there exists a Mitchell-Steel model $L[\mathbb{E}]$ for a proper class of measurable Woodin cardinals within which $\mathbb{E}$ is definable (even from parameters).
- Suppose $L[\mathbb{E}]$ is a Mitchell-Steel model within which there exists a Woodin cardinal. The inductive first order requirements on $L_\alpha[\mathbb{E}]$ are very complicated:
  - things only get worse for the generalized Mitchel-Steel models.

Two questions

1. *Is there a simple candidate for the axiom "$V = \mathrm{Ultimate}\text{-}L$"?*
2. *Is $\mathrm{Ultimate}\text{-}L$ even a good partial-extender model?*

Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}$ is *universally* Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}$, the preimage of A by π has the property of Baire in the space Ω.

- Universally Baire sets are an abstract generalization of the borel sets.

Theorem

Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire. Then every set

$$B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$$

is universally Baire.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and large cardinal axioms

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

- $\Theta^{L(A,\mathbb{R})}$ is a measure of the complexity of A.

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and the Inner Model Program

Theorem (Steel)

Suppose that there is a proper class of Woodin cardinals and let $\delta = \Theta^{L(\mathbb{R})}$.

Then $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ is a Mitchell-Steel model.

Theorem

Suppose that there is a proper class of Woodin cardinals.

*Then $\mathrm{HOD}^{L(\mathbb{R})}$ is **not** a Mitchell-Steel model.*

There is another class of solutions to the inner model problem for large cardinals.

- *strategic partial-extender models*
- *previously unknown.*

The axiom for $V = \mathrm{Ultimate}\text{-}L$

(meta) Conjecture: The axiom for $V = \mathrm{Ultimate}\text{-}L$

- *There is a strong cardinal and a proper class of Woodin cardinals.*
- *For each Σ_3-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \models \varphi$$

  where $\Theta = \Theta^{L(A,\mathbb{R})}$.

- This axiom settles (modulo axioms of infinity) *all* sentences about $\mathcal{P}(\mathbb{R})$ (and much more) which have been shown to be independent by Cohen's method.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

The Continuum Hypothesis holds.

More consequences of $V = \mathrm{Ultimate}\text{-}L$

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

For each cardinal κ, if $V[G]$ is a set-generic extension of V then there exists an elementary embedding

$$\pi : (H(\kappa^+))^V \to N$$

such that $(\pi, N) \in V$ and such that $N \in \mathrm{HOD}^{V[G]}$.

Corollary ($V = \mathrm{Ultimate}\text{-}L$)

$V = \mathrm{HOD}$.

Corollary ($V = \mathrm{Ultimate}\text{-}L$)

V is the minimum universe of the Generic-Multiverse.

The proof that $V = \text{Ultimate-}L$ implies $V = \mathrm{HOD}$

Fix a regular uncountable cardinal κ.

- There is an elementary embedding $\pi : H(\kappa^+) \to N$ such that $N \in \mathrm{HOD}$.

Let $\langle S_\alpha : \alpha < \kappa \rangle$ be a partition of the set $\{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \omega\}$ into stationary sets and let

$$\langle T_\beta : \beta < \pi(\kappa) \rangle = \pi(\langle S_\alpha : \alpha < \kappa \rangle).$$

Define

$$Z = \{\beta < \pi(\kappa) \mid T_\beta \cap C \neq \emptyset \text{ for all closed cofinal sets } C \subset \sup(\pi[\kappa])\}$$

and for each $X \in N$, let $X^* = \{\alpha < \kappa \mid \pi(\alpha) \in X\}$.
First key point:

- $Z \in \mathrm{HOD}$ (since $N \in \mathrm{HOD}$) and $Z = \pi[\kappa]$.

Second key point:

- $\{X^* \mid X \in N\} \subset \mathrm{HOD}$ (since $\pi[\kappa] \in \mathrm{HOD}$) and this implies that $\mathcal{P}(\kappa) \subset \mathrm{HOD}$ since $\mathcal{P}(\kappa) = \{X^* \mid X \in N\}$.

The Ultimate-L Conjecture

Ultimate-L Conjecture

(ZFC) *Suppose that δ is an extendible cardinal. Then there is a transitive class N such that:*

1. *N is a weak extender model for δ is supercompact.*
2. *$N \subseteq \mathrm{HOD}$.*
3. *$N \models$ "$V = \mathrm{Ultimate}\text{-}L$".*

- Ultimate-L Conjecture if true would show that there is no generalization of Scott's Theorem to Ultimate-L.