General approach to Ramsey theory and a new Ramsey theorem

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Outline of Topics

1. Introduction

2. Dual Ramsey theorem for trees

3. Algebraic notions

4. Abstract Ramsey and abstract pigeonhole statements

Introduction

For $n \in \mathbb{N}$, put

$$[n] = \{1, 2, \ldots, n\};$$

in particular,

$$[0] = \emptyset.$$

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Later, close connections between extreme amenability and finite Ramsey theory were discovered.

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Kechris–Pestov–Todorcevic: G is extremely amenable if and only if $G = \mathrm{Aut}(A)$ for a countable ultra-homogeneous structure A such that

— A is linearly ordered;

— the class of all finite substructures of A is Ramsey.

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A **copy** of S in T is the image of S under an embedding from S to T.

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monochromatic.

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Prömel–Voigt: Rigid surjections from $[n]$ to $[m]$ are in a bijective correspondence with m-partitions of n:

$$s \to \mathcal{P}_s = \{s^{-1}(i)\colon i \in [m]\}.$$

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D. Bartošova–A. Kwiatkowska: a Ramsey statement needed for dynamics (computation of the universal minimal flow) of the homeomorphism group of the Lelek fan

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s is **dual to** i if for each $w \in T$

$$s([w]) = i^{-1}([w]).$$

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In the case when S and T are linear orders $[k]$ and $[l]$, the new notion of rigid surjection coincides with the old one.

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Algebraic notions

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Most finite unstructured Ramsey theorems are special instances of the above theorem. The theorem also makes it possible to prove new results.

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Normed composition spaces

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- $|\cdot|$ is a function from A to a partially ordered set $(L, \leq)$ (**norm**).

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(iii) if $|b| \leq |c|$ and $a \cdot c$ is defined, then so is $a \cdot b$ and $|a \cdot b| \leq |a \cdot c|$.

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where i is the injection of f.

Let $w \in T$, $f : T^w \to S$ and $g : V \to T$. Define

$$g \cdot f = f \circ g^w.$$

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otherwise, let v be the second $\leq_S$-largest vertex in S, and let

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A **with** $\cdot$**,** ∂**,** $|\cdot|$ **defined above is a normed composition space.**

Lifting multiplication to sets

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$$F \bullet G = F \cdot G.$$

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$\mathcal{F}$ is a family of sets over A.

Abstract Ramsey and abstract pigeonhole statements

Ramsey statement

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$\mathcal{F}$ a family over a normed composition space

The following condition is our Ramsey statement:

(R) given $d > 0$, for each $P \in \mathcal{F}$,
there is $F \in \mathcal{F}$ such that
$F \bullet P$ is defined and
for every d-coloring of $F \bullet P$ there is $f \in F$ such that $f \cdot P$ is monochromatic.

Dual trees (ctd)

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Condition (R) is the dual Ramsey theorem for trees with **sealed** rigid surjections.

Pigeonhole statement

$a \in A$ can be viewed as a partial function from A to A defined on

$$\{x \in A \colon a \cdot x \text{ defined}\}.$$

a is a **restriction of** $b \in A$ if b extends a as a partial function.

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Price: make f behave as prescribed by some $a \in A$ on a part of A

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Main Abstract Theorem

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Let $\mathcal{F}$ be a family over a normed composition space. Assume $\mathcal{F}$ fulfills conditions (A), (B), and (C). If for each $F \in \mathcal{F}$ there is $t \in \mathbb{N}$ with $\partial^t F$ having one element, then (P) implies (R).

Dual trees (ctd)

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$\mathcal{F}$ fulfills conditions (A)–(C).

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So (R) holds for $\mathcal{F}$ by Main Abstract Theorem.

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Lemma

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So dual Ramsey theorem for trees **holds**.