# The Many Lives of Generalized Quantifiers

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# On a generalization of quantifiers

by

A. Mostowski (Warszawa)

In this paper I shall deal with operators which represent a natural generalization of the logical quantifiers[1]. I shall formulate, for the generalized quantifiers, problems which correspond to the classical problems of the first-order logic. Some of these problems will be solved in the present paper, other more interesting ones are left open.

Most of our discussion centers around the problem whether it is possible to set up a formal calculus which would enable us to prove all true propositions involving the new quantifiers. Although this problem is not solved in its full generality, yet it is clear from the partial results

# Summary

- Mostowski introduced generalized quantifiers in 1957.
- Generalized quantifiers have thrived in
  - logic,
  - linguistics,
  - computer science.
- They manifest the perfect symbiosis between model theory and set theory.
- Dependence logic is a new way of thinking about generalized quantifiers with potential applications in many areas.

# Generalized quantifiers

**Definition (Mostowski 1957)**

A *generalized quantifier* (in Mostowski's sense) is a class $Q$ of structures $\mathfrak{M} = \langle I, A \rangle$, where $A \subseteq I$, such that

$$[\mathfrak{M} \in Q \wedge \mathfrak{M} \cong \mathfrak{M}'] \Rightarrow \mathfrak{M}' \in Q.$$

This can be turned into a logical operation:

$$\mathfrak{M} \models \mathrm{Q}x\phi(x, \vec{a}) \iff \langle M, \{a : \mathfrak{M} \models \phi(a)\} \rangle \in Q.$$

Denote the extension of first order logic by $L(\vec{Q})$.

# Examples of generalized quantifiers

**Example**

1. $\exists = \{\langle I, A\rangle \ : \ A \neq \emptyset\}$
2. $\forall = \{\langle I, A\rangle \ : \ A = I\}$
3. $\exists_n = \{\langle I, A\rangle \ : \ A \geq n\}$ (Counting quantifier)
4. $Q_\alpha = \{\langle I, A\rangle \ : \ |A| \geq \aleph_\alpha\}$

# Central questions

- Axiomatizability?
- Löwenheim-Skolem, compactness, interpolation theorems?
- Hierarchies of definability?

**Theorem (Mostowski '57)**

*Let $Q_1, \ldots, Q_n$ be quantifiers on $\aleph_0$. The following conditions are equivalent:*

1. *The set of valid sentences of $L(Q_1, \ldots, Q_n)$ is r.e.*
2. *Each $Q_i$ is a Boolean combination of counting quantifiers.*

**Theorem (Mostowski '57)**

*The following condition characterises first order logic among logics of the form $L(\vec{Q})$: Every sentence with an infinite model has models of all infinite cardinalities.*

# Subsequent results

- $L(Q_1)$ is recursively axiomatizable (Vaught '64)
- OPEN: $L(Q_2)$ recursively axiomatizable?
- But GCH implies $L(Q_{\alpha+1})$ is recursively axiomatizable (Chang '65).
- Consistently $L(Q_1, Q_2)$ not countably compact. (Shelah '05). [This does not decide rec. axiomatizability.]

# Curious situation

- “More than there are **natural** numbers": axiomatizable.
- “At least as many as there are **natural** numbers": non-axiomatizable.
- “More than there are **real** numbers": axiomatizable.
- “At least as many as there are **real** numbers": OPEN.

# Shelah's method of identities

**Theorem (Shelah-V. '06)**

*Suppose* $\kappa_0, \ldots, \kappa_{m-1}$ *is a sequence of uncountable cardinals. The following conditions are equivalent:*

1. *A certain canonical set of sentences is a complete axiomatization of* $L(\exists^{\geq\kappa_n})_{n<m}$.
2. $L(\exists^{\geq\kappa_n})_{n<m}$ *is* $\lambda$*-compact for all* $\lambda < \min\{\kappa_0, \ldots, \kappa_{m-1}\}$.
3. *There is a "fundamental* $(\kappa_n)_{n<m}$*-pattern".*

# Lindström quantifiers

## Definition (Lindström 1966)

A *generalized quantifier* (in Lindström's sense) is a class $Q$ of structures $\mathfrak{M}$ of a fixed vocabulary, such that

$$[\mathfrak{M} \in Q \wedge \mathfrak{M} \cong \mathfrak{M}'] \Rightarrow \mathfrak{M}' \in Q.$$

Logical operation:

$$\mathfrak{M} \models \mathrm{Q}\vec{x}\phi(\vec{x}, \vec{a}) \iff \langle M, \{\vec{b} : \mathfrak{M} \models \phi(\vec{b}, \vec{a})\}\rangle \in Q.$$

Denote the extension of first order logic by $L(\vec{Q})$.

## Example

1. $I = \{\langle I, A, B\rangle \ : \ |A| = |B|\}$ (Härtig-quantifier)
2. $W = \{\langle I, <\rangle \ : <$ well-orders $I\}$
3. $\begin{pmatrix} \forall & \exists \\ \forall & \exists \end{pmatrix} = \{\langle I, A\rangle \ : \exists f, g \forall x \forall y (x, y, f(x), g(y)) \in A\}$ (Henkin quantifier)
4. $\bigwedge_{\alpha<\beta} \phi_\alpha, (\exists x_\alpha)_{\alpha<\beta}\phi$ (Infinitary logic)
5. $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \bigwedge_n \phi_n(x_0, y_0, \ldots, x_n, y_n)$ (Game quantifier)
6. $TC = \{\langle A, E, X, Y\rangle : \langle A, E\rangle$ is a graph and from every $x \in X$ there is a path in the graph to some $y \in Y\}$ (Transitive closure)
7. Non-example: Second order logic.

## Example

1. Magidor-Malitz-quantifier: “There is an uncountable set homogeneous for $\phi(x, y)$. Axiomatizable assuming $\diamondsuit$ (Magidor-Malitz ’77)
2. Cofinality quantifier: “$\phi(x, y)$ defines a linear order of cofinality $\aleph_0$". Axiomatizable and fully compact (Shelah ’75)

# Lindström extends Mostowski's results

**Theorem (Lindström '66)**

*Let $Q_1, \ldots, Q_n$ be quantifiers such that $L(\vec{Q})$ has the downward Löwenheim-Skolem Theorem down to $\aleph_0$. The following conditions are equivalent:*

1. *The set of valid sentences of $L(Q_1, \ldots, Q_n)$ is r.e.*
2. *Each $Q_i$ is first order definable.*

**Theorem (Lindström '66)**

*The following condition characterises first order logic among logics of the form $L(\vec{Q})$: Every sentence with an infinite model has models of all infinite cardinalities.*

Independently by Harvey Friedman 1970.

## Definition (Mostowski '68, Lindström '69)

An abstract logic is a class $L^*$ of classes such that

- Each class in $L^*$ is an isomorphism-closed class of $L$-structures from some vocabulary $L$.
- $L^*$ includes first order definable model classes and is closed under renaming, Booleans and first order quantification.

## Example

- $L(\vec{Q})$, $L_{\kappa\lambda}$, $L^2$
- Non-examples: Models with a metric, a topology, a measure, etc. Intuitionistic logic, modal logic, etc.

**Theorem (Lindström '69)**

*Let $L^*$ be an abstract logic with "effective syntax" such that $L^*$ has the downward Löwenheim-Skolem Theorem down to $\aleph_0$. The following conditions are equivalent:*

1. *The set of valid sentences of $L^*$ is r.e.*
2. *Each sentence in $L^*$ is first order definable.*

**Theorem (Lindström '69)**

*The following condition characterises first order logic among logics of the form $L^*$: Every sentence with an infinite model has models of all infinite cardinalities.*

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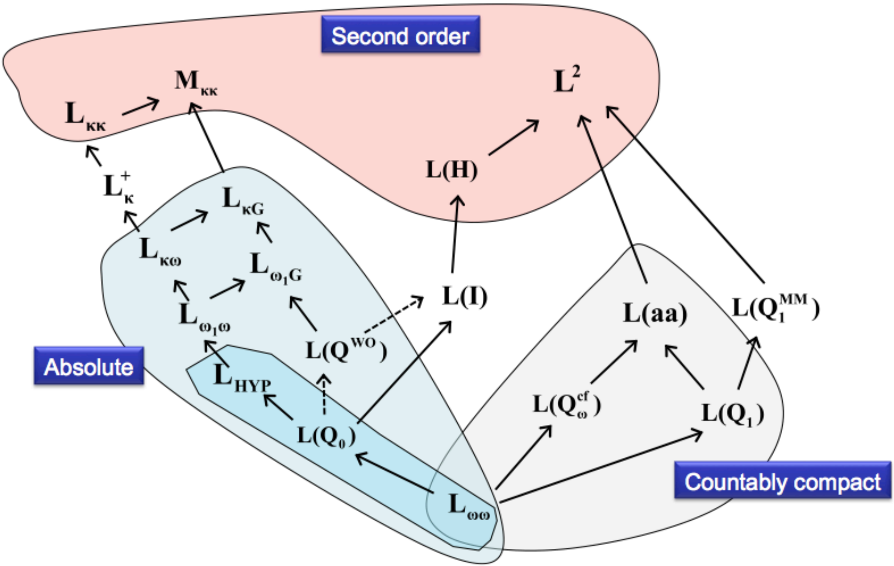
Second order
$M_{\kappa\kappa}$
$L_{\kappa\kappa}$
$L^2$
$L^+_\kappa$
$L(H)$
$L_{\kappa G}$
$L_{\kappa\omega}$
$L_{\omega_1 G}$
$L(I)$
$L_{\omega_1\omega}$
$L(aa)$
$L(Q_1^{MM})$
Absolute
$L(Q^{WO})$
$L_{HYP}$
$L(Q^{cf}_\omega)$
$L(Q_1)$
$L(Q_0)$
Countably compact
$L_{\omega\omega}$

# A recent Lindström-type result

**Theorem (Shelah '12)**

*Suppose $\kappa = \beth_\kappa$. There is a logic $L^+_\kappa$ such that:*

- $L_{\kappa\omega} \leq L^+_\kappa \leq L_{\kappa\kappa}$.
- *$L^+_\kappa$ has a Lindström-type[a] characterisation.*
- *$L^+_\kappa$ satisfies the Craig Interpolation Theorem.*

[a] A combination of downward LS-theorem and undefinability of well-order.

## Theorem (Mostowski '68)

*A sufficient condition for a logic to fail to satisfy the Craig Interpolation Theorem is that $(\mathbb{N}, +, \cdot)$ is characterisable in the logic and there is a recursive bound on the Borel rank of definable classes of models of the form $(\mathbb{N}, +, \cdot, R)$.*

## Example

$L(Q_0)$, $L^2_w$ (different versions)

# The beautiful $\Delta$-operation

## Definition

The abstract logic $\Delta(L^*)$ consists of model classes that are projections and co-projections of $L^*$-definable model classes.

## Examples

- $\Delta(L_{\omega\omega}) = L_{\omega\omega}$. (Craig '57)
- $\Delta(L_{\omega_1\omega}) = L_{\omega_1\omega}$. (Lopez-Escobar '65)
- $\Delta(L(Q_0)) = L_{HYP}$ (Barwise, Friedman, '72)

## Theorem (Shelah-V. *to appear*)

*If CH, then* $\Delta(L(Q_1)) \neq L(Q_1, \ldots, Q_n)$ *for all Lindström-quantifiers* $Q_1, \ldots, Q_n$.

# Definability: A symbiosis between model theory and set theory

A predicate $R$ of set theory and an abstract logic $L^*$ are symbiotic if:

1. Every sentence in $\Delta(L^*)$ defines a $\Delta_1(R)$-class of models.
2. Every $\Delta_1(R)$-class of models is definable in $\Delta(L^*)$.

**Example**

- $L^2$ and $x = \mathcal{P}(y)$.
- $L(I)$ and $|\alpha| = \alpha$.
- $L(W)$ and $x = x$.

**Theorem (Strong form of Mostowski's result)**

*If $L^*$ is symbiotic with $R$ and absolute relative to $R$, then $L^* \neq \Delta(L^*)$.*

# Symbiosis of model theory and set theory I

| | | |
|---|---|---|
| $L_{\omega\omega}$ | $\Delta_1^{KPU^-}$ | (Wilmers), Akkanen '95 |
| $L(Q_0)$-def. | Borel of rank $< \omega$ | Mostowski '68 |
| $\Delta(L(Q_0)$-def. | Borel of rank $< \omega_1^{CK}$ | Mostowski '68 |
| $L_{HYP}$-def. | $\Delta_1^{KP}$ | Barwise '74 |
| $L_{\omega_1\omega}$-def. | Borel | Lopez-Escobar '65, Scott '64 |
| $\Sigma_1^1(L_{\omega_1\omega})$-def. | Analytic | Vaught '71 |
| $\Sigma_1^1(L_{\omega_1 G})$-def. | $\Sigma_2^1$ | Vaught '71 |

# Symbiosis of model theory and set theory II

| $\Delta(L(Q_0)$-definable | Borel of rank $< \omega_1^{CK}$ | Mostowski '68 |
|---|---|---|
| $\Delta(L(W))$-def. | $\Delta_1$ | (Many people '77) |
| $\Delta(L^2)$-def. | $\Delta_2$ | V. '77 |
| $\Delta(L^*)$-def. | $\Delta_1(R)$ | V. '77 |
| Löwenheim n. | $\sup\{\Pi_1(R)\text{-def.}\}$ | Krawczyk-Marek '77 V. '77 |
| Hanf number | $\sup\{\Sigma_1(R)\text{-def.}\}$ | Krawczyk-Marek '77, V. '77 |
| Decision prob. | $\Pi_1(R)$-complete | V. '77 |
| Compactness th. | Large cardinals | |
| Löwenheim-Sk. | Reflection princ. | |

# Symbiosis of model theory and set theory III

| $\Delta(L(Q_0)$-definable | Borel of rank $< \omega_1^{CK}$ | Mostowski '68 |
|---|---|---|
| $\Delta(L_{\omega\omega})$ | Recursive | (Trakhtenbrot '50) |
| $\Delta^1_1(L_{\omega\omega})$ | $NP \cap co - NP$ | Fagin '74 |
| $FP$(on ordered) | PTIME | Immerman '82, Vardi '82 |
| $L_{\omega\omega}$(on ordered) | $\subseteq$ LOGSPACE | |

# The world of natural language quantifiers is rich

- Most boys run.
  $Qxy\phi(x)\psi(y)$.
  Monotone unary quantifier.
- Most girls in my class know each other.
  $Ram^2(Q)xyz\phi(x)\psi(y,z)$.
  Ramsey-lift of the unary $Q$.
- Most neighbours like each other.
  $Res^2(Q)xuv\phi(x)\psi(u,v)$.
  Resumption (or vectorization) of the unary $Q$.

# The goal

The goal is to classify natural language quantifiers.

Find a “basis" in terms of which every other quantifier is expressible.

# Hierarchy theorems

## Theorem

1. *(Hella-V-Westerståhl '97) For non-trivial Q, $Ram^{k+1}(Q)$ is not definable in $L_{\infty\omega}(\mathbf{Q}_k)$, where $\mathbf{Q}_k$ is the class of all k-ary quantifiers.*
2. *(Hella-V-Westerståhl '97) For non-trivial Q, $Res^{k+1}(Q)$ is not definable in $L_{\infty\omega}(\mathbf{Q}_1)$.*
3. *(Hella-Luosto-V '96) There is a binary (PTIME) Q which is not definable in $L_{\omega\omega}(Ram^{<\omega}(\mathbf{Q}_1))$.*

Note: If for all $m$ there is an $m+1$-ary (PTIME) $Q$ which is not definable in $L_{\omega\omega}(Res^{<\omega}(\mathbf{Q}_m))$, then $P \neq NP$.

# Computer science

$$TC = \{\langle A, E, X, Y\rangle : \langle A, E\rangle \text{ is a graph}$$
$$\text{and from every } x \in X \text{ there is a path in the graph}$$
$$\text{to some } y \in Y\}$$

$$ATC = \{(A, E, X, Y) : (A, E) \text{ is a graph}, X \subseteq A, Y \subseteq A$$
$$\text{and every } x_0 \in X \text{ has a neighbour } x_1$$
$$\text{whose every neighbour } x_2 \text{ has a neighbour } x_3$$
$$\text{etc ... until we reach a } y \in Y\}$$

**Theorem**

1. *(Immerman '87) NLOGSPACE* $= L_{\omega\omega}(Res^{<\omega}(TC))$
2. *(Dahlhaus '87) FP* $= L_{\omega\omega}(Res^{<\omega}(ATC))$.

OPEN: Such a result for PTIME?

# Generalized quantifiers are not the solution

**Theorem (Hella '96)**

*On unordered finite models, PTIME is not the extension of fixed point logic by finitely many generalized quantifiers.*

# Dependence logic - generalized quantifiers in a new way

The goal is to find a common logic behind the various uses of dependence and independence in different areas of science and humanities.

# The technical tool behind dependence logic

$$S(\mathfrak{M}, \phi, s) \iff \mathfrak{M} \models_s \phi,$$

where $s$ is an assignment.

$$\Downarrow$$

$$S(\mathfrak{M}, \phi, X) \iff \mathfrak{M} \models_X \phi,$$

where $X$ is a set of assignments.

# Team semantics

Single assignments

$$\Downarrow$$

Sets of assignments

$$||$$

Teams

# Team semantics

One assignment

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $s$ | 1 | 0 | 2 |

# Team semantics

| One assignment | |
|---|---|
| <table><tr><th></th><th>$x$</th><th>$y$</th><th>$z$</th></tr><tr><td>$s$</td><td>1</td><td>0</td><td>2</td></tr></table> | |
| <table><tr><th></th><th>**color**</th><th>**shape**</th><th>**height**</th></tr><tr><td>$s$</td><td>yellow</td><td>wrinkled</td><td>tall</td></tr></table> | |

# Team semantics

| One assignment | Team |
|---|---|
| | |

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $s$ | 1 | 0 | 2 |

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $s_1$ | 1 | 0 | 2 |
| $s_2$ | 2 | 1 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s_n$ | 1 | 3 | 1 |

| | **color** | **shape** | **height** |
|---|---|---|---|
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# Team semantics

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|---|---|

**One assignment**

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|---|---|---|---|
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**One assignment**

| | **color** | **shape** | **height** |
|---|---|---|---|
| $s$ | yellow | wrinkled | tall |

**Team**

| | **color** | **shape** | **height** |
|---|---|---|---|
| $s_1$ | yellow | wrinkled | tall |
| $s_2$ | green | wrinkled | short |
| $s_3$ | green | round | tall |

Suppose $Q$ is a Lindström quantifier of type $(n)$. We add a new atomic formula $\mathcal{Q}(x_1, \ldots, x_n)$ with the interpretation

$$M \models_X \mathcal{Q}(x_1, \ldots, x_n) \iff (M, X_{x_1,\ldots,x_n}) \in Q,$$

where

$$X_{x_1,\ldots,x_n} = \{s \restriction \{x_1, \ldots, x_n\} \ : \ s \in X\}.$$

## Example

- $Q^D_{x_i,x_j} = \{(M, X) \ : \ X \subseteq M^n$ and $X_{x_i,x_j}$ is a function$\}$. (Functional dependence)
- $Q^I_{x_i,x_j} = \{(M, X) \ : \ X \subseteq M^n$ and $X_{x_i,x_j}$ is of the form $A \times B$ for some $A$ and $B\}$ (Independence)
- $Q^{\subseteq}_{x_i,x_j} = \{(M, X) \ : \ X \subseteq M^n$ and $X_{x_i} \subseteq X_{x_j}\}$ (Inclusion)

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# First order logic with dependence atoms

- Take the least $\kappa$ such that if a sentence has a model $\mathfrak{M}$, then it has a model $\mathfrak{N} \subseteq \mathfrak{M}$ of cardinality $< \kappa$. It is $\aleph_1$.
- Take the least $\kappa$ such that if a sentence avoids a model $\mathfrak{M}$, then it avoids a model $\mathfrak{N} \subseteq \mathfrak{M}$ of cardinality $< \kappa$. It is the first supercompact cardinal.

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- Intuitionistic implication: $X$ satisfies $\phi \supset \psi$ iff for all $Y \subseteq X$, $Y \models \phi$ implies $Y \models \psi$.
- Galois-connection:

$$\phi \wedge \psi \models \theta \text{ iff } \phi \models \psi \supset \theta.$$

- Fan Yang '12: Full second order power.

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# Kangas-Hyttinen-V. '11

## Theorem

*Suppose that $\kappa$ is a regular cardinal such that $\kappa = \aleph_\alpha$, $\beth_{\omega_1}(|\alpha| + \omega) \leq \kappa$ and $2^\lambda < 2^\kappa$ for all $\lambda < \kappa$. Let $T$ be a countable complete first order theory. Then TFAE:*

1. *Every model of $T$ of size $\kappa$ is characterizable, up to isomorphism, by a sentence of (infinitary) dependence logic with intuitionistic implication.*
2. *$T$ is a shallow, superstable theory without DOP or OTOP.*

**Theorem**

1. *Dependence logic is downward closed NP. (Kontinen-V. '09)*
2. *Independence logic is NP. (Galliani '12)*
3. *Inclusion logic is FP. (Galliani-Hella '13)*

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