Transcendence, differential equations, and model theory.
Mostowski Centenary Meeting

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Introduction I

- I will give a leisurely introduction to and treatment of some applications of logic to transcendence properties of solutions of ODE's, and will be telling a story in a sense.
- The ODE's we have in mind are usually of the form $P(y^{(n)}, ..., y'', y', y) = 0$ where P is a polynomial over $\mathbb{C}(t)$, and one is interested in solutions $y = y(t)$ meromorphic on some disc in $\mathbb{C}$.
- Among the abstract environments for studying such objects is Kolchin's DAG, "differential algebraic geometry", bearing the same relation to differential polynomial equations above that algebraic geometry bears to polynomial equations.
- Solution sets are viewed as abstract point sets in some differential field (F, ∂) containing the ground field.

Introduction II

- To the left of DAG (but closely related) is the model theory of differentially closed fields of characteristic 0.
- The relevant first order theory DCF_0 is the model companion of the theory (or class) of differential fields of characteristic 0 in the language $L = \{+, \times, 0, 1, \partial\}$.
- DCF_0 is decidable (as well as having QE in L and being ω-stable) so one might wonder if such a "tame" first order theory can be at all relevant to the chaotic world of ODE's.
- To the right of the Kolchin theory are algebraic geometric/differential geometric approaches and formalisms for ODE's and PDE's (foliations, connections, D-modules) and further to the right integrable systems, physics,...

Introduction III

- One does not really expect DAG or DCF_0 to say much about describing actual solutions of a given ODE, but rather to describe properties of solutions and sets of solutions, and corresponding invariants of the equations.
- A basic example is differential Galois theory:
- Let $Y' = AY$ be a linear differential equation over $\mathbb{C}(t)$ in vector form. Then the set $V \subset F^n$ of solutions (in the differential field F of meromorphic functions on some disc, or even abstract solutions in a differential closure F of $\mathbb{C}(t)$) is an n-dimensional vector space over the field $\mathbb{C}$ of constants.
- Let $Y_1, .., Y_n$ be a basis of V, a so-called fundamental system of solutions.
- Then $tr.deg(\mathbb{C}(t)(Y_1, .., Y_n)/\mathbb{C}(t))$ is a basic invariant of the equation, and equals the dimension of its differential Galois group, an algebraic subgroup of $GL_n(\mathbb{C})$.

Introduction IV

- Describing and/or computing these invariants (the Galois group and its dimension), the inverse differential Galois problem, as well as generalizations to other base fields and more general differential equations supporting a Galois theory, are problems which are in the domain of DAG and DCF_0.
- DCF_0 also contributes the tools and machinery of stability theory, which are meaningful and will be touched on below.

The Painlevé equations I

- Painlevé (and contemporaries at the beginning of the 20th century) wanted to classify or describe those second order ODE's of the form $y'' = f(y, y', t)$ which share a certain tameness property with linear ODE's (the Painlevé property) and which are "irreducible" in the sense that no solution can be expressed in terms of "classical or known" functions.
- Following Painlevé's "Lecons de Stockholm", Umemura in the 1980's gave a precise inductive definition of "classical function", involving algebraic functions, solutions of linear differential equations and more generally logarithmic differential equations on complex abelian varieties, and solutions of order 1 equations.

The Painlevé equations II

- The Painlevé property concerns the analytic continuation of local solutions and doesn't appear to be a differential algebraic or model theoretic property except in order 1 where there is a classification. (Give definition?)
- In any case, Painlevé and others gave a list of six families $P_I - P_{VI}$ of equations $y'' = f(y, y')$ over $\mathbb{C}(t)$, claiming that up to some notion of equivalence the general equations in the various families exhaust the "irreducible" such equations.
- For example, P_I is the single equation $y'' = 6y^2 + t$, and P_{II} the one-parameter family $y'' = 2y^3 + ty + \alpha$, $\alpha \in \mathbb{C}$.
- There were various claims regarding the irreducibility of general equations in the families, but only with the work of the Japanese school in the past 30 years has this been carried out in a rigorous fashion.

The Painlevé equations III

- Umemura isolated a property which he called property (H) and which I call "Umemura-irreducibility": $y'' = f(y, y')$ is Umemura irreducible if for any differential fields $\mathbb{C}(t) \subseteq K \subseteq F$ and solution $y \in F$, either $y \in K^{alg}$ (y is algebraic over K) or y, y' are algebraically independent over K (y is a generic solution over K).
- And he noted that the equation is irreducible if it has no solution algebraic over $\mathbb{C}(t)$ AND is Umemura-irreducible.
- A trivial but important observation is that Umemura-irreducibility of the equation means precisely strong minimality of the set X defined by the equation in an ambient differentially closed field.
- Where a definable set in an ambient structure M is strongly minimal if X is infinite and every subset of X definable in M is finite or cofinite.

The Painlevé equations IV

- An enormous amount of work has been done in the past 30 years on describing for each of the families $P_I - P_{VI}$ those parameters in the parameter space for which the corresponding equation is Umemura-irreducible and/or has an algebraic solution.
- Essentially, for generic $(\alpha, \beta, \ldots)$ in the parameter space, $y'' = f_{(\alpha,\beta,..)}(y, y')$ is Umemura-irreducible and has no algebraic solutions, BUT the set of parameters for which the equation is U-reducible is Zariski-dense, as well as the set of parameters for which the equation has an algebraic solution.
- For example $P_{II}(\alpha)$ is U-reducible (so of Morley rank > 1) iff $\alpha \in \mathbb{Z} + 1/2$, and has an algebraic solution (which is moreover unique) iff $\alpha \in \mathbb{Z}$.

Algebraic independence of solutions and their derivatives I

- A strong version of irreducibility of $y'' = f(y, y')$ is that for any distinct solutions $y_1, .., y_n$, $y_1, y_1', ..., y_n, y_n'$ are algebraically independent over $\mathbb{C}(t)$. We call this *very strong irreducibility*.
- It says that any permutation of the set of solutions in a differentially closed field is elementary over $\mathbb{C}(t)$. It implies strong minimality of the solution set, no algebraic over $\mathbb{C}(t)$ solutions, as well as ω-categoricity of the solution set (in a suitable sense).
- So very strong irreducibility of the "general" equations in each of the Painlevé families is a kind of culmination of the Painlevé program.
- Nishioka proved very strong irreducibility of the single equation P_I.

Algebraic independence of solutions and their derivatives II

Theorem 0.1

The generic equation in each of the families P_{II}, P_{IV}, P_V is very strongly irreducible. The generic equation in families P_{III} and P_{VI} is at least ω-categorical.

- We discuss the proof.
- The first step, valid for each of the families $P_{II} - P_{VI}$, says that any algebraic dependence between $y_1, y_1', ..., y_n, y_n'$ over $\mathbb{C}(t)$ is witnessed pairwise, namely by y_i, y_i', y_j, y_j' for some $1 \leq i < j \leq n$.
- This uses the deepest results in DCF_0, namely a coarse classification of strongly minimal sets in DCF_0, and will be omitted.
- The second step is elementary model theory and I can and will describe it here, concentrating on P_{II}.

Algebraic independence of solutions and their derivatives III

- Fix a generic equation $y'' = 2y^3 + ty + \alpha$ in P_{II}, namely $\alpha \in \mathbb{C}$ is transcendental.
- Write this equation as a (qf) formula $\phi(y, t, \alpha)$ of L.
- By Step 1, we assume towards a contradiction that there are solutions $y \neq z$ such that y, y', z, z' are algebraically dependent over $\mathbb{C}(t)$.
- Express this by an L-formula $\psi(y, z, c, \alpha, t)$ where c is a tuple from $\mathbb{C}$.
- By strong minimality and the lack of algebraic solutions, any two solutions have the same type over $\mathbb{C}(t)$.
- Hence $\forall y(\phi(y, \alpha, t) \rightarrow \exists c \exists z(c' = 0 \wedge \phi(z, \alpha, t) \wedge y \neq z \wedge \psi(y, z, c, \alpha, t)))$ holds in our given differential closure F of $\mathbb{C}(t)$.

Algebraic independence of solutions and their derivatives IV

- Let us denote the above sentence (with parameters α, t) by $\sigma(\alpha, t)$.
- By QE in DCF_0, σ is equivalent to a quantifier-free L-formula with parameters α, t. And by transcendentality of $\alpha \in \mathbb{C}$, $\sigma(\beta, t)$ is true for cofinitely many constants $\beta \in \mathbb{C}$.
- In particular $\models \sigma(n, t)$ for some $n \in \mathbb{Z}$.
- But then choosing y to be the unique algebraic solution of $y'' = 2y^3 + ty + n$, we obtain another solution z, algebraic over $\mathbb{C}(t)(y, y')$ so also over $\mathbb{C}(t)$, contradicting the fact above that the equation has a unique algebraic solution.
- Difficult to see any other proof than this using the given data. END.