Large scale geometry of metrisable groups

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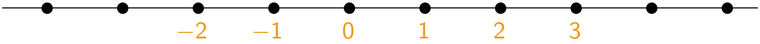
where $g \in \Gamma$ and $s \in \Sigma \setminus \{1\}$.

The resulting graph $\mathrm{Cayley}(\Gamma, \Sigma)$ is connected and hence Γ is a metric space, when given the shortest-path metric ρ_Σ.

Consider first $(\mathbb{Z}, +)$ with generating set $\Sigma = \{-1, 1\}$.



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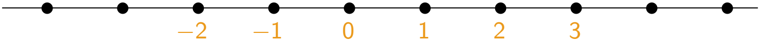
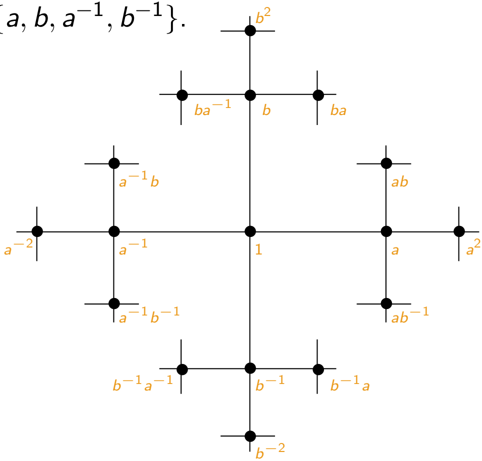


Similarly, let $\mathbb{F}_2$ be the free non-abelian group on generators a, b and set $\Sigma = \{a, b, a^{-1}, b^{-1}\}$.



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ρ_Σ is called the word metric induced by the generating set Σ.

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Taking N to be the largest n needed for the finitely many $s' \in \Sigma'$, one sees that

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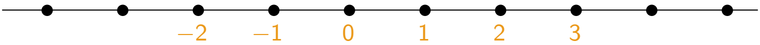

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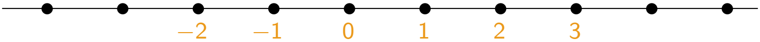
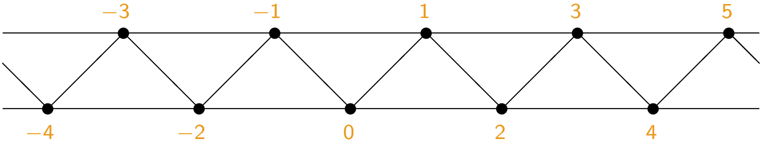


Whereas, with generating set $\Sigma = \{-2, -1, 1, 2\}$, we have



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Instead of finite generation, one should now consider compact generation.

Moreover, we should relax the notion of bi-Lipschitz equivalence in order to get canonical metrics.

Definition

A map $F\colon (X,d)\to(Y,\partial)$ between metric spaces is said to be a *quasi-isometric embedding* *if there are constants K, C so that*

$$\frac{1}{K}\cdot d(x,y)-C \;\leqslant\; \partial(Fx,Fy) \;\leqslant\; K\cdot d(x,y)+C.$$

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Moreover, F *is a quasi-isometry if, in addition, its image is cobounded, meaning that*

$$\sup_{y\in Y} \partial(y, F[X]) < \infty.$$

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As an aside, we should mention that by work of Gleason, G carries canonical small scale geometry if and only if G is a Lie group.

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So, as opposed to quasi-isometry, the interdependence of R and S is no longer affine.

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Then

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(a) How should we define $g_n \to \infty$ in a general metrisable group?
(b) Is there a reasonable definition of proper metrics on G?
(c) Is there a canonical large scale or coarse geometry on such G?

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This agrees with the usual definition in locally compact groups.

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Definition

A compatible left-invariant metric d on a metrisable group G is said to be metrically proper if $d(g_n, 1) \to \infty$ whenever $g_n \to \infty$.

Relative property (OB)

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Also, the group G is said to have property (OB) if it has property (OB) relative to itself.

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A subset A of a metrisable group G is said to have property (OB) relative to G if, for every compatible left-invariant metric d on G, one has

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Also, the group G is said to have property (OB) if it has property (OB) relative to itself.

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Also, by the existence of proper metrics, in locally compact groups, the relative property (OB) coincides with relative compactness.

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We should mention that this fails without the assumption of separability.

Also, the metrically proper compatible left-invariant metric d is unique up to coarse equivalence and thus defines the coarse geometry of G.

Maximal (or word) metrics

Having characterised the separable metrisable groups admitting canonical metrics up to coarse equivalence, we turn to the finer issue of characterising those admitting canonical metrics up to quasi-isometry.

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$$\partial(g,f)\leqslant K\cdot d(g,f)+C.$$

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If it exists, we can therefore talk unambigously of the quasi-isometric structure of G.

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A metric space (X, d) is said to be large scale geodesic if there is $K \geqslant 1$ so that, for all $x, y \in X$, there are $z_0 = x, z_1, z_2, \ldots, z_n = y$ satisfying

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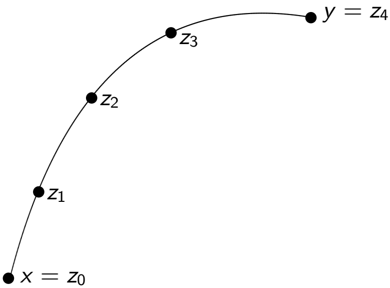

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is a quasi-isometry between (G, ∂) and (X, d).

Large scale geometry of first order structures and their automorphism groups

As mentioned earlier, by a lemma due to Cameron, the automorphism group $\mathrm{Aut}(\mathbf{K})$ of a countable $\aleph_0$-categorical structure $\mathbf{K}$ has property (OB) and therefore admits a compatible left-invariant metric d maximal for large distances.

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Nevertheless, there are plenty of more interesting examples.

But to identify these, it will be useful to have some verifiable criteria for admitting metrically proper or maximal metrics.

Non-Archimedean Polish groups

Recall that a Polish group is called non-Archimedean if it admits a neighbourhood basis at 1 consisting of open subgroups or, equivalently, if it is isomorphic to a closed subgroup of S_∞.

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Also, for a closed subgroup $G \leqslant S_\infty = \mathrm{Sym}(\mathbb{N})$, there is a countable relational language L and an ultrahomogeneous L-structure $\mathbf{K}$ with universe $\mathbb{N}$ so that

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So, when studying non-Archimedean Polish groups, we may always represent them as automorphism groups of ultrahomogeneous relational structures.

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By a theorem of Fraïssé, there is a correspondence between countable ultrahomogeneous L-structures $\mathbf{K}$ and Fraïssé classes $\mathcal{K}$ given by

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Here, $\mathbf{K}$ is uniquely determined up to isomorphism by $\mathcal{K} = \mathrm{Age}(\mathbf{K})$ and we say that $\mathbf{K}$ is the Fraïssé limit of $\mathcal{K}$.

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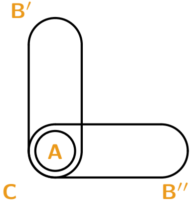
   - $(*)$ *for every $\mathbf{B} \in \mathcal{K}$ containing $\mathbf{A}$, there are $n \geqslant 1$ and a finite family $\mathcal{S} \subseteq \mathcal{K}$ so that, for all $\mathbf{C} \in \mathcal{K}$ and embeddings $\eta_1, \eta_2 \colon \mathbf{B} \hookrightarrow \mathbf{C}$ with $\eta_1|_{\mathbf{A}} = \eta_2|_{\mathbf{A}}$, one can find some $\mathbf{D} \in \mathcal{K}$ containing $\mathbf{C}$ and a path $\mathbf{B}_0 = \eta_1\mathbf{B}, \mathbf{B}_1, \ldots, \mathbf{B}_n = \eta_2\mathbf{B}$ of isomorphic copies of $\mathbf{B}$ inside $\mathbf{D}$ with $\langle \mathbf{B}_i \cup \mathbf{B}_{i+1} \rangle \in \mathcal{S}$,*

To visualise (2)($*$), let $\mathbf{C}$ be an amalgam of two copies $\mathbf{B}'$ and $\mathbf{B}''$ of $\mathbf{B}$ over $\mathbf{A}$.



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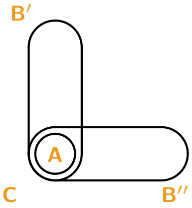

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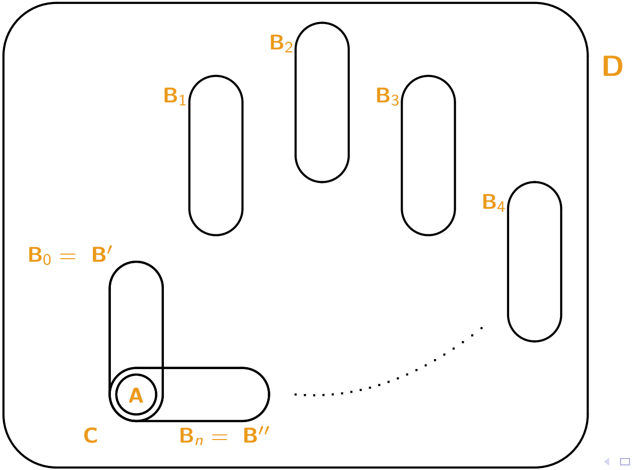

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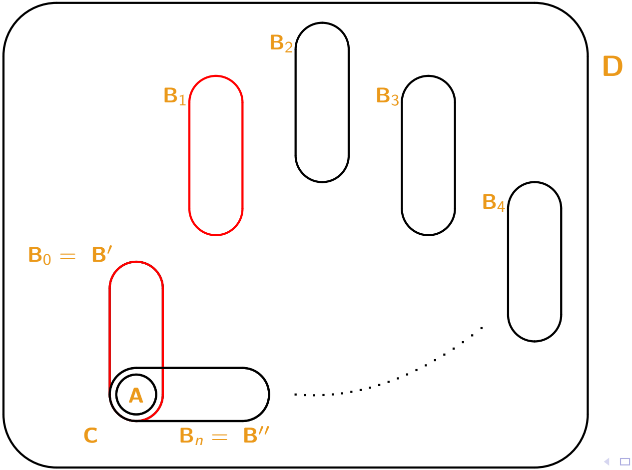

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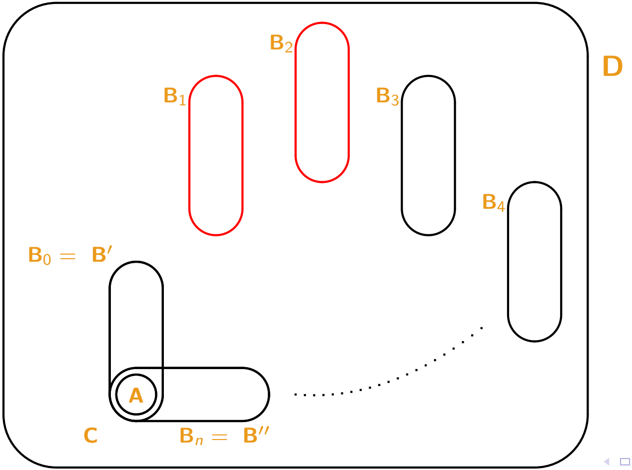

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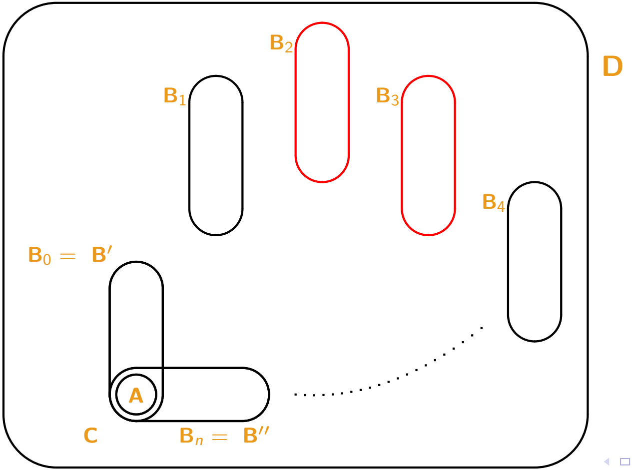

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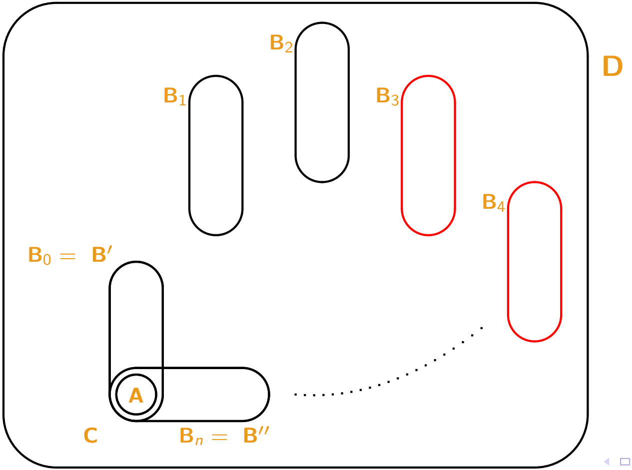

Theorem (Existence)

Suppose $\mathcal{K}$ is a Fraïssé class of finite structures with Fraïssé limit $\mathbf{K}$. Then the following are equivalent.

1. $\mathrm{Aut}(\mathbf{K})$ *admits a compatible left-invariant metric maximal for large distances,*
2. *there is $\mathbf{A} \in \mathcal{K}$ satisfying the following two conditions*
   (i) *for every $\mathbf{B} \in \mathcal{K}$ containing $\mathbf{A}$, there are $n \geqslant 1$ and a finite family $\mathcal{S} \subseteq \mathcal{K}$ so that, for all $\mathbf{C} \in \mathcal{K}$ and embeddings $\eta_1, \eta_2 \colon \mathbf{B} \hookrightarrow \mathbf{C}$ with $\eta_1|_{\mathbf{A}} = \eta_2|_{\mathbf{A}}$, one can find some $\mathbf{D} \in \mathcal{K}$ containing $\mathbf{C}$ and a path $\mathbf{B}_0 = \eta_1\mathbf{B}, \mathbf{B}_1, \ldots, \mathbf{B}_n = \eta_2\mathbf{B}$ of isomorphic copies of $\mathbf{B}$ inside $\mathbf{D}$ with $\langle \mathbf{B}_i \cup \mathbf{B}_{i+1} \rangle \in \mathcal{S}$,*

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   - (ii) *there is a finite family $\mathcal{R} \subseteq \mathcal{K}$ so that, for all $\mathbf{B} \in \mathcal{K}$ and isomorphic copies $\mathbf{A}', \mathbf{A}'' \subseteq \mathbf{B}$ of $\mathbf{A}$, there is some $\mathbf{C} \in \mathcal{K}$ containing $\mathbf{B}$ and isomorphic copies $\mathbf{A}_0 = \mathbf{A}', \mathbf{A}_1, \ldots, \mathbf{A}_m = \mathbf{A}'' \subseteq \mathbf{C}$ of $\mathbf{A}$ so that $\langle \mathbf{A}_i \cup \mathbf{A}_{i+1} \rangle \in \mathcal{R}$.*

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Moreover, if $\mathbf{A}$ *and* $\mathcal{R}$ *are as in (2), let* X *denote the set of isomorphic copies of* $\mathbf{A}$ *in* $\mathbf{K}$ *and put*

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Then (X, E) *is a connected graph and the mapping*

$$g \in \mathrm{Aut}(\mathbf{K}) \mapsto g \cdot \mathbf{A} \in X$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{K})$ *and* X *equipped with its path metric.*

Example: The rational Urysohn space

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Let $\mathcal{S} \subseteq \mathcal{U}$ consist of the isometry class of $\mathbf{B} \oplus_{2\delta} \mathbf{B}$ and put $n = 2$.

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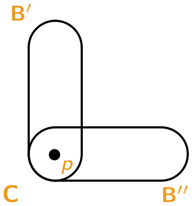

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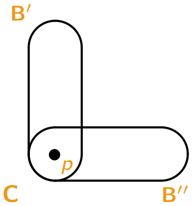

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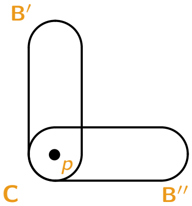

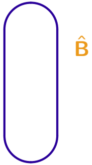

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Thus, $\langle \mathbf{B}' \cup \hat{\mathbf{B}} \rangle \cong \mathbf{B} \oplus_{2\delta} \mathbf{B} \in \mathcal{S}$ and $\langle \hat{\mathbf{B}} \cup \mathbf{B}'' \rangle \cong \mathbf{B} \oplus_{2\delta} \mathbf{B} \in \mathcal{S}$.

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We remark that $n = 2$ is minimal, since there are infinitely many distinct ways of amalgamating two copies of **B** over the single point p (provided, of course, that **B** is non-trivial).

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$$\rho(x, y) \leqslant k \quad \Leftrightarrow \quad \exists z_0 = x, z_1, z_2, \ldots, z_k = y \colon d(z_i, z_{i+1}) = 1.$$

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However, since $\mathbb{QU}$ is not quasi-isometric to $T_{\aleph_0}$, it follows that

$$\mathrm{Isom}(\mathbb{QU}) \ncong \mathrm{Aut}(T_{\aleph_0}).$$