New examples of small Polish structures

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Abstract

We answer some questions from [4] by giving suitable examples of small Polish structures. First, we present a class of small Polish group structures without generic elements. Next, we construct a first example of a small non-zero-dimensional Polish G-group.

0 Introduction

In [4], Krupiński defined and investigated Polish structures by methods motivated by model theory.

Definition 0.1. *A Polish structure is a pair (X, G), where G is a Polish group acting faithfully on a set X so that the stabilizers of all singletons are closed subgroups of G. We say that (X, G) is small if for every $n < \omega$, there are only countably many orbits on X^n under the action of G.*

A particularly interesting situation is when the underlying set X is a group itself. Throughout this paper, we follow the terminology from [4].

Definition 0.2. *Let G be a Polish group.*
(i) A Polish group structure is a Polish structure (H, G) such that H is a group and G acts as a group of automorphisms of H.
(ii) A (topological) G-group is a Polish group structure (H, G) such that H is a topological group and the action of G on H is continuous.
(iii) A Polish [compact] G-group is a topological G-group (H, G), where H is a Polish [compact] group.

Let (X, G) be a Polish structure. For any finite $C \subseteq X$, by G_C we denote the pointwise stabilizer of C in G, and for a finite tuple a of elements of X, by $o(a/C)$ we denote the orbit of a under the action of G_C (and we call it the orbit of a over C).

A fundamental concept for [4] is the relation of nm-independence in an arbitrary Polish structure.

Definition 0.3. *Let a be a finite tuple and A, B finite subsets of X. Let $\pi_A : G_A \to o(a/A)$ be defined by $\pi_A(g) = ga$. We say that a is nm-independent from B over A (written $a \perp^{nm}_A B$) if $\pi_A^{-1}[o(a/AB)]$ is non-meager in $\pi_A^{-1}[o(a/A)]$. Otherwise, we say that a is nm-dependent on B over A (written $a \not\perp^{nm}_A B$).*

This is a generalization of m-independence, which was introduced by Newelski for profinite structures ([6, 7]). Under the assumption of smallness, nm-independence has similar properties to those of forking independence in stable theories, and hence it allows to transfer some ideas and techniques from stability theory to small Polish structures (which are purely topological objects). The investigation of Polish structures has been undertaken in [5] and [1]. For example, in [5], some structural theorems about compact G-groups were proved, and in [1], dendrites were considered as Polish structures, and some properties introduced in [4] were examined for them.

The class of Polish structures contains many more interesting examples from classical mathematics than the class of profinite structures. For example, for any compact metric space P, if we consider the group $Homeo(P)$ of all homeomorphisms of P equipped with the compact-open topology, then $(P, Homeo(P))$ is a Polish structure (examples of small Polish structures of this form were investigated in [4] and [1]). Also, if H is a compact metrizable group, then $(H, Aut(H))$ is a Polish group structure. However, in the class of small Polish group structures, it is more difficult to construct interesting examples. In the present paper, we answer some questions from [4] by constructing suitable examples of small Polish group structures.

The following is [4, Question 5.4] (see Definition 1.3 for the notion of nm-generic orbit).

Question 0.4. *Let (H, G) be a small Polish group structure. Does H possess an nm-generic orbit?*

Proposition 5.5 from [4] gives us a positive answer to Question 0.4 in the class of small Polish G-groups. In Section 2, we construct a class of small Polish group structures for which the answer to Question 0.4 is negative.

The following problem was formulated in [4] (after Question 5.32):

Problem 0.5. *Find a non-zero-dimensional, small Polish G-group.*

In Section 3, we construct a small Polish G-group (H, G), such that H is homeomorphic to the complete Erdös space, which is known to be one-dimensional.

1 Preliminaries

If A is a finite subset of X (where (X, G) is a Polish structure), we define the algebraic closure of A (written $Acl(A)$) as the set of all elements of X with countable orbits over A. If A is infinite, we define $Acl(A) = \bigcup\{Acl(A_0) : A_0 \subseteq A \text{ is finite}\}$. By Theorems 2.5 and 2.10 from [4], we have:

Theorem 1.1. *In any Polish structure (X, G), nm-independence has the following properties:*
(0) (Invariance) $a \perp^{nm}_A B \iff g(a) \perp^{nm}_{g[A]} g[B]$ whenever $g \in G$ and $a, A, B \subseteq X$ are finite.
(1) (Symmetry) $a \perp^{nm}_C b \iff b \perp^{nm}_C a$ for every finite $a, b, C \subseteq X$.
(2) (Transitivity) $a \perp^{nm}_B C$ and $a \perp^{nm}_A B$ iff $a \perp^{nm}_A C$ for every finite $A \subseteq B \subseteq C \subseteq X$ and $a \subseteq X$.
(3) For every finite $A \subseteq X$, $a \in Acl(A)$ iff for all finite $B \subseteq X$ we have $a \perp^{nm}_A B$.
If additionally (X, G) is small, then we also have:
(4) (Existence of nm-independent extensions) For all finite $a \subseteq X$ and $A \subseteq B \subseteq X$ there is $b \in o(a/A)$ such that $b \perp^{nm}_A B$.

By [4, 2.14], under some assumptions, nm-dependence in a G-group (H, G) can be expressed in terms of the topology on H:

Theorem 1.2. *Let (X, G) be a Polish structure such that G acts continuously on a Hausdorff space X. Let $a, A, B \subseteq X$ be finite. Assume that $o(a/A)$ is non-meager in its relative topology. Then, $a \perp^{nm}_A B \iff o(a/AB) \subseteq_{nm} o(a/A)$.*

Counterparts of various notions from model theory were studied by Krupiński in the context of Polish structures. One of them is the notion of a generic orbit:

Definition 1.3. *Let (H, G) be a Polish group structure. We say that the orbit $o(a/A)$ is left nm-generic (or that a is left nm-generic over A) if for all $b \in H$ with $a \perp^{nm}_A b$, one has that $b \cdot a \perp^{nm} A, b$. We say that it is right nm-generic if, for b as above, we have $a \cdot b \perp^{nm} A, b$. An orbit is nm-generic if it is both right and left nm-generic.*

It was noticed in [4] that nm-generics have similar properties to generics in simple theories, e.g. being right nm-generic is equivalent to being left nm-generic. We recall Proposition 5.5 from [4], which gives us a positive answer to Question 0.4 for the class of small G-groups (H, G) in which H in not meager in itself (this holds, for example, in all Polish G-groups).

Fact 1.4. *Suppose (H, G) is a small G-group. Assume H is not meager in itself (e.g. H is Polish or compact, or, more generally, Baire). Then, at least one nm-generic orbit in H exists, and an orbit is nm-generic in H iff it is non-meager in H.*

2 Small Polish group structures without generic elements

In this section, we construct a class of small Polish group structures for which the answer to Question 0.4 is negative.

Suppose (X, G) is a Polish structure. Let H be an arbitrary group. For any $x \in X$ we consider an isomorphic copy $H_x = \{h_x : h \in H\}$ of H. By $H(X)$ we will denote the group $\bigoplus_{x \in X} H_x$. Although $H(X)$ is not necessarily commutative, we will denote its group action by $+$. For any $y \in H(X)$ there are $h_1, \dots, h_n \in H \backslash \{e\}$ and pairwise distinct $x_1, \dots, x_n \in X$ such that $y = (h_1)_{x_1} + \dots + (h_n)_{x_n}$. Then, by $\tilde{y}$ we will denote the set $\{x_1, \dots, x_n\}$. We also put $\tilde{A} = \bigcup_{y \in A} \tilde{y}$ for any $A \subseteq H(X)$.

The group G acts as automorphisms on $H(X)$ by

$$g((h_1)_{x_1} + \dots + (h_n)_{x_n}) = (h_1)_{gx_1} + \dots + (h_n)_{gx_n}.$$

It is easy to see that if $h_1, \dots, h_k \in H \backslash \{e\}$ are pairwise distinct, and $x_{1,1}, \dots, x_{1,i_1}, x_{2,1}, \dots, x_{2,i_2}, \dots, x_{k,1}, \dots, x_{k,i_k} \in X$ are pairwise distinct as well, then the stabilizer of $(h_1)_{x_{1,1}} + \dots + (h_1)_{x_{1,i_1}} + \dots + (h_k)_{x_{k,1}} + \dots + (h_k)_{x_{k,i_k}}$ consists exactly of those elements of G which stabilise each of the finite sets $\{x_{i,j} : j = 1, \dots, i_j\}$. Thus, we get that for every $a \in H(X)$, G_a is a subgroup of finite index in $G_{\tilde{a}}$, and hence, for every finite $A \subseteq H(X)$, G_A is a subgroup of finite index in $G_{\tilde{A}}$. It is now not difficult to notice the following:

Proposition 2.1. *If (X, G) is a Polish structure, and H is a group, then $(H(X), G)$ is a Polish group structure. If, additionally, (X, G) is small and H is countable, then $(H(X), G)$ is small.*

Also, one can describe nm-independence on $H(X)$ in terms of nm-indepenence on X:

Proposition 2.2. *Let (X, G) be a Polish structure, and H a countable group. Then, for any finite $A, B, C \subseteq H(X)$, we have:*

$$A \perp^{nm}_C B \iff \tilde{A} \perp^{nm}_{\tilde{C}} \tilde{B}.$$

The following corollary gives us a negative answer to Question 0.4 in its full generality, i.e., in the class of all Polish group structures. Recall that Fact 1.4 tells us that the answer is positive for small Polish G-groups.

Corollary 2.3. *Let (X, G) be a small Polish structure, where X is uncountable. If H is a nontrivial countable group, then $(H(X), G)$ is a small Polish group structure, and it has no generic orbit (neither left nor right).*

Proof. Take any $a \in H(X)$ and a finite $A \subseteq H(X)$. We will show that $o(a/A)$ is not a generic orbit. Take any $h \in H \backslash \{e\}$ and $b \in X \backslash Acl(\emptyset)$ such that $b \perp^{nm} \widetilde{A, a}$. Then, by Proposition 2.2, $h_b \perp^{nm}_A a$. Since $b \perp^{nm} \tilde{A}, \tilde{a}$ and $b \notin Acl(\emptyset)$, we see that $b \notin \tilde{a}$. Hence, $\widetilde{a + h_b} = \tilde{a} \cup \{b\}$. But $\tilde{a}, b \not\perp^{nm} b$, so, again by Proposition 2.2, we have that $a + h_b \not\perp^{nm} h_b$. Hence, $o(a/A)$ is not a generic orbit. □

By the above corollary and Fact 1.4, we get in particular:

Corollary 2.4. *If (X, G) is an uncountable small Polish structure, and H is a nontrivial countable group, then there is no Polish topology on $H(X)$ such that the action of G on $H(X)$ is continuous.*

Now, we will give a variant of the above construction. Suppose R is a countable commutative ring, and (X, G) is a small Polish structure. Let $R(X) = R[(y_x)_{x \in X}]$ be the ring of polynomials in variables $(y_x)_{x \in X}$ with coefficients in R. Then G acts on $R(X)$ by $gw(y_{x_1}, \dots, y_{x_n}) = w(y_{gx_1}, \dots, y_{gx_n})$. If R is a countable field, we can additionally consider $R(X)_0 = R((y_x)_{x \in X})$, the field of rational functions in variables $(y_x)_{x \in X}$ with coefficients in R. Then, G acts on $R(X)$ by $gf(y_{x_1}, \dots, y_{x_n}) = f(y_{gx_1}, \dots, y_{gx_n})$. As for $H(X)$, one can check that $(R(X), G)$, $(R_0(X), G)$ are small Polish structures. As above, on can show that these structures (which we could call Polish ring structures and Polish field structures) have no generics (in the sense of the additive group), and hence, there is no Polish topology on $R(X)$ or on $R(X)_0$ such that the action of G is continuous.

3 A non-zero-dimensional small Polish G-group

In this section, we construct a first example of a small non-zero-dimensional Polish G-group.

We define a structure of a group on the complete Erdös space as in [3, Proposition 4.3] (with the only difference that we do not choose a particular p), which is done as follows. Fix any $p \in [1, \infty)$. We let $C \subseteq \mathbb{R}$ be the ternary Cantor set, and $X = C^\omega \cap \ell^p$. By results of [2] and [3], X (considered with the topology induced from ℓ^p) is homeomorphic to the complete Erdös space. Consider the standard bijection $\phi : 2^\omega \to C$ and the product map $\psi := \phi^\omega : (2^\omega)^\omega \to C^\omega$. It follows exactly as in [3, Proposition 4.3] that $H := \psi^{-1}[X]$ is a subgroup of $(2^\omega)^\omega$ (we will identify the latter group with $2^{\omega \times \omega}$ in the natural way), and becomes a Polish group with the topology induced from X by ψ (and is homeomorphic to the complete Erdös space). This topology is generated by the norm $||z|| := ||\psi(z)||_p$, $z \in H$. We also put $||z|| = \infty$ if $z \in 2^{\omega \times \omega} \backslash H$. For a subset A of $\omega \times \omega$, we define $||A|| := ||\chi_A||$, where χ_A is the characteristic function of A.

Now, we will define an action of a Polish group G on H. Let G_1 be the group of all permutations of $\omega \times \omega$. For any $g \in G_1$, we define the support of g to be $supp(g) = \{a \in \omega \times \omega : g(a) \neq a\}$. We put:

$$G = \{g \in G_1 : ||supp(g)|| < \infty\} < G_1.$$

It is clear that for any $g \in G$ and $h \in H$, the composition $h \circ g : \omega \times \omega \to 2$ is an element of H (since $||h \circ g|| \leq ||h|| + ||supp(g)||$). Hence, we can define an action of G on H by $gh = h \circ g^{-1}$. Then, G acts on H as automorphisms (both algebraic and topological). Notice, however, that if we consider G with the product topology, then this action is not continuous. Hence, we need another topology on G.

We define a metric d on G:

$$d(f, g) = ||supp(f^{-1}g)||.$$

We consider G with the topology generated by d. In this way, we obtain (the first known) example of a small, non-zero-dimensional Polish G-group (in this poster we skip the proof):

Proposition 3.1. *G is a Polish group and (H, G) is a small, Polish G-group.*

One can check that $\mathcal{NM}$-rank of every uncountable 1-orbit in (H, G) is equal to ∞.

Question 3.2. *Is there an nm-stable, non-zero-dimensional small Polish G-group?*

Notice that since the product $H \times H$ is homeomorphic to H, we cannot obtain examples of higher dimensions just by taking finite cartesian powers of H.

Question 3.3. *Is there a small Polish G-group of dimension greater than one?*

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